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Collateral Re-use, Liquidity and Financial Stability

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ABSTRACT

This work provides a model where the repercussions on financial stability of collateral *re-use* in repo contracts can be analysed and assessed. In the model, the rationale for repo contracts is the arbitrage activity of a leveraged hedge fund, which is profitably financed by a dealer bank. Repo contracts, in connection with collateral *re-use*, lubricate both the credit and the financial system, increasing the financial operators' profits and improving equilibrium rates and volumes. At the same time, they amplify the leverage of the whole economy, making it vulnerable to shocks. Introducing a default risk for the hedge fund, the proposed model identifies diverging effects of collateral *re-use* on financial stability. In states with low dealer bank profitability, the increase in collateral *re-use* renders a sound dealer bank management style the profit maximising strategy. Where an unsound balance sheet expansion is highly profitable, the increase in collateral *re-use* provides destabilising incentives to the dealer bank.

Keywords: repo markets, collateral re-use, rehypothecation, systemic risk

JEL codes: E58, G01, G21, G23

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This work aims at contributing to the existing literature on repurchase agreements (repo) by providing a model that can both flexibly incorporate the main important features of the repo market and, at the same time, enable its assessment in terms of financial stability. In repo contracts, a seller sells a security to a buyer while agreeing to buy it back from him on a predefined future date for a predefined price. Essentially, repos can be regarded as a form of secured borrowing, where a security is used as collateral, that is as a tool to mitigate credit risk. An additional feature of repos is the possibility for the buyer (the cash lender) to use collateral in further trades (re-use). Thanks to their versatility, repo trades lie at the crossroads between a market for funds and a market for securities. In particular, the funding and the hedging motives underlying repo contracts can be overshadowed by the search for securities and by the trading motive (see, among others, Duffie 1996).

Despite being a market for secured lending, repo markets are not at all alien to financial fragility. Repo markets have been involved in the first stages of the Financial Crisis in 2007-09 and the ensuing reforms of this market segment have radically changed it. According to Gorton and Metrick (2012) repo markets are vulnerable to runs that can severely impair the liquidity and solvency of market participants. Haircuts (or margins) limit the leverage levels of participants in the repo markets, by allowing the borrowing of cash for less than 100 per cent of the value of the collateral. In case of increased perceived risk in the financial markets, an increase in haircuts can also become the driver of market dry-ups, since it can end up hindering market's access for holders of risky collateral, correspondingly reducing the overall level of liquidity available in the markets.

Repo contracts involve the possibility of collateral re-use by the cash lender. This means that the security involved in the repo contract can be pledged by the lender who wants to enter another analogous repo contract. This generates the possibility of repo chains. Repo chains affect the market in two ways. On the one hand, they mitigate collateral scarcity and increase liquidity in securities market, thus increasing efficiency in the securities' market. On the other hand, repo chains increase the interconnections in the financial system, allowing for wider and quicker transmissions of shocks. Moreover, the so-called "collateral multiplier" effect, that makes possible that a single security underpins a multiplicity of lending relationship via re-use, increases the extent in which the financial system is affected by fluctuations in the collateral value.

The proposed model aims at incorporating the main traits of repo markets, including the heterogeneity of the economic agents involved, who have to satisfy different kinds of needs. On the one hand, the funding motive is a characteristic of institutional sectors who access the secured loans market in order to satisfy liquidity needs or to profit from the credit intermediation activity. On the other hand, for economic agents active in the trading of securities (such as hedge funds or speculators), repo contracts typically sustain leveraged investment profiles and arbitrage trading. Dealer banks are typically interposed between the two parts and feature large gross exposures in their balance sheets, while totalling limited net exposures.

The proposed model also aims at providing an encompassing treatment for both the liquidity shocks in the financial markets and the liquidity shocks in the credit and deposits market. The dealer bank bridges with its balance sheet between two dimensions: the dimension of the liquidity insurance provided to demand depositors (basically by means of its maturity transformation) and the dimension of the liquidity provision in the financial markets made possible by its financing of the arbitrageurs. Keeping these two dimensions together makes it possible to analyse how shocks on the treasury department of the dealer bank affect its overall business including its commercial bank department and in particular its funding channels.

For the above mentioned reasons, the model features several specialised financial operators. In the middle of the model, the dealer bank is modelled having regard for its funding sources, both short term unsecured and longer term secured, and for its maturity transformation activity. The profit generating activity of the dealer bank is confined to the secured financing of a hedge fund, active as arbitrageur in the financial market. The funding of the hedge fund takes place in an oligopoly market dominated by the dealer bank itself. The model features different types of financial contracts, covering spot trades, repo contracts, securities lending and demand deposits.

One of the original features of the proposed model is the possibility of connecting the lending and the funding side of the dealer bank. The dealer bank's funding is shown to be affected by the securities' market conditions and by the shocks taking place in the repo market. These features are reflected in the capability of the model of triggering classic bank runs depending on the business model (and risk profile) selected by the bank in order to operate in the money market. Beside its pivotal role in the repo market, based on its capacity of connecting holders of securities and cash to users of both, the dealer bank incorporates traits of the classic banking literature (for reference, see, Bryant 1980, Diamond and Dybvig 1983 and Allen and Gale 1998).

The conclusions attained by using the proposed model are relevant for both the literature on money markets and for the literature on financial stability. The main novelty of the model resides in its capability of highlighting two different and interrelated channels for the collateral re-use to affect financial stability. Instead of resorting to fire-sales, the model exploits the effects in terms of leverage and profitability of collateral re-use. The conclusions introduce a new set of testable hypotheses in the field. The theoretical analysis implies, in fact, that the effect of collateral re-use on financial stability is state-dependent: in low profitability contexts collateral re-use can improve financial stability despite increasing leverage, thanks to the stabilising effect on profits. Where large gains are available by increasing in an exaggerated way the bank balance sheet, there collateral re-use deploys its negative effects.

The present work builds on existing contributions in the economic modelling of repo markets. In line with Brunnermeier and Pedersen (2009) and the later Ranaldo et al. (2016) repo market is modelled as a conjunction point between securities trading and cash lending, used as source of funding by leveraged arbitrageurs speculating on securities prices' deviations from fundamentals. Differently from these works, unbalances in repo markets take place as a consequence of defaults and depreciations and not because of dry-ups and fire sales. Similarly to Gottardi et. al. (2017) and to Andolfatto et al. (2017), re-use is seen as a mean to increase market liquidity. The re-use of collateral is explicitly analysed, but in contrast with these works, the proposed model directly analyses the implications for financial stability inherent to the re-use of the collateral. As in Antinolfi et al. (2015), the re-use produces a trade-off for the policy maker, but differently from the mentioned authors, this take place at a structural (business model) level. A shock to the collateral quality reminiscent of those in Gorton and Ordoñez (2014) is used in order to model the trigger for the systemic crisis. Similarly to Infante (2015) and Eren (2014), the proposed model features a set of heterogeneous and specialised economic agents which get connected by means of repo chains. Differently from the two mentioned models, the risk of defaults and contagion originates in the securities' market and is propagated through the financial system by the repo chains. The work is also connected to Brumm et al. (2018), where an extended analysis of the policy implications of regulating collateral re-use is conducted. In comparison with this last paper, the present work focuses more on the micro-structure of the repo market and offers an analysis of the policy implications of collateral re-use more oriented to contagion risk and state-dependence. Finally, the modelling of the dealer bank relationship with the unsecured demand depositors has taken inspiration from Martin et al. (2010), where the short-term demand deposits financing the dealer bank activity are the mechanism at the basis of "repo runs".

I. Re-use in the repo market

The collateral *re-use* in the repo market is a concept strictly connected to that of *re-hypothecation*. Both concepts are very specific for the repo market and in a way contribute to distinguish the repo market from the other secured markets (such as for instance the mortgage market). Thanks to *re-use* (or *re-hypothecation*) the security pledged as collateral by a debtor in a repo contract becomes, for all possible purposes, a property of the lender. As such it can be used and pledged again in a subsequent separate contract (from which the term *re-use*). As clarified by ICMA (2019), re-use is "the onward outright sale of collateral by a repo buyer to a third party in the cash market". As we can read in Comotto (2014), from a legal point of view, especially in Europe, the two concepts or *re-use* and *re-hypothecation* can be considered identical. In the US, this is true because of a particular property of the repo contracts: the "automatic stay exemption". This exemption enables the cash lender in a repo transaction to liquidate the collateral in case of counterparty insolvency, in order to be restored of his losses. A specific right of re-use associated with the "automatic stay exemption", renders the US concept of *re-hypothecation* analogous to the more EU related concept of collateral *re-use*.

Among the interesting properties of repo markets, the collateral *re-use* makes what are called "repo chains" possible. A "repo chain" is a sequence of repo contracts where the same security is used as collateral subsequently by different operators (in addition, it is also possible that the same operator occurs more times in the chain). The existence and magnitude of these "chains" has been investigated and treated empirically by several authors employing different techniques. A first influential approach was proposed by Singh and Aitken (2009) and makes use of balance-sheet data. The empirical methodology employed by the authors is to examine the yearly balance sheet of the major international "dealer banks" and compare the amount of collateral deposited by primary collateral providers, such as funds and commercial banks and the amount of repo contracts outstanding at the same date employing that collateral. The authors' work rely on the pivotal intermediation activity of a limited number of banks, among which Singh (2011) in an other work mentions "Goldman Sachs, Morgan Stanley, JP Morgan, BoA/Merrill and Citibank in the U.S. In Europe and elsewhere [...] Deutsche Bank, UBS, Barclays, Credit Suisse, Société Générale, BNP Paribas, HSBC, Royal Bank of Scotland and Nomura".

The possibility to use multiple times the same collateral for financing purpose generates what is called a "collateral multiplier". In market conditions discouraging or impeding unsecured lending, the fluent working of the credit market crucially depends on the availability to the debtors of quality collateral to pledge in repo transactions. The "collateral multiplier" identifies the proportion between the collateral available and the financing extended. It can be argued, like Singh (2011) does, that in order to correctly estimate the monetary aggregates, the central banks should include the pledged collateral in the calculation. In fact changes in the collateral "velocity" due to (among others) availability of quality collateral, for instance, could reduce, *ceteris paribus*, the amount of credit available in the economy.

According to Singh et al. (2018), the historical evolution of the collateral velocity features a severe drop in correspondence to the great financial crisis of 2007-09. Singh (2011) indicates a drop from a level of 3 for the velocity in 2007 to a level of 2.4 in 2010. This evolution, in combination with an overall decrease in collateral deposited by primary collateral providers (from a global amount of 1,695 billion dollars in 2007 to 1,119 billion dollars in 2010), indicates a sharp negative trend for the repo market and for the collateral re-use over the crisis years. A trend, though, that could be possibly reverted in present times, according to Singh et al. (2018), who register how collateral velocity has rebound during the 2017 (2.0, from the all times low of 2016: 1.8).

It must be noted that the empirical assessment of collateral re-use and of collateral multiplier varies across studies and estimates. On the one hand, in a work employing survey data, ESRB (2014) is able to provide, for the sole European repo market, an estimate of collateral re-use roughly in line with the ones of Singh and coauthors (re-use factor is estimated as equal to 2). On the other hand, works employing granular transaction by transaction data, tend to produce far lower estimates for collateral re-use and multiplier (see, among others, Fuhrer et al. 2015, Ferrari et al. 2017). Instead, the metrics employed by the empirical studies on the subject tend to be relatively similar, even if it is foreseeable that, over time, with the increase of information and data on the topic, more refined and sophisticated quantification techniques will emerge (see, for a systematic review of metrics, FSB 2017b).

II. The model

A. Framework

The economy consists of four distinct agents (or categories of agents): a continuum of depositors, a hedge fund (HF), a dealer bank (DB) and a continuum of other financial intermediaries (OFI). In the economy two different types of financial assets are bought/sold or borrowed/lent: cash and a security. Each agent has an infinite time horizon. Time is discrete and extends over an infinite sequence of periods $t_1, t_2 \ldots$. A diagram illustrating the general structure of the economy is provided in Fig. 1. In this diagram, the arrows connecting the agents illustrate the type of exchange taking place between them: the *violet* arrow connecting the HF to the financial market indicates that the HF buys/sells the security against cash (spot trades). The *blue* arrows connecting HF, DB and OFI indicate that between these three agents repo trades take place (a secured lending contract). Repo trades are assumed to take place with a 0 haircut (this assumption could appear extreme, but it is the case in several real settings - see Fuhrer et al. 2015 for reference to the Swiss repo market). The *orange* arrow connecting the depositors to the DB represents a standard deposit contract (an unsecured lending contract). In the following sections financial assets and market participants will be dealt with in detail.

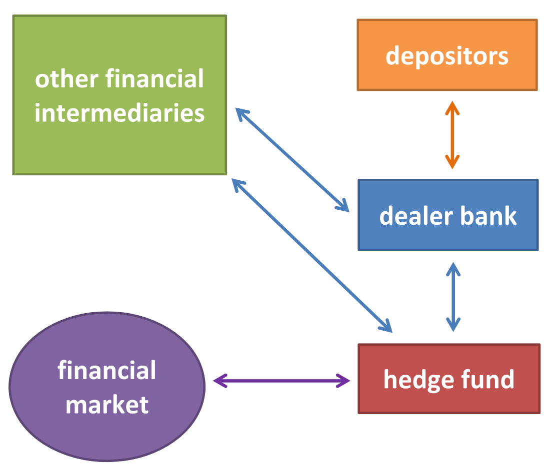

Figure 1. Market structure

A.1. Financial assets

Two financial assets are available in the economy to investors. The returns of each category of financial asset has its own specific expected value and variability. The first financial asset is cash. Cash represents a storing technology, that enables to transfer consumption from one period to the other, but also between different sub-periods belonging to the same period. Its return is certain and equal to $r_c = 0$.

The second financial asset is a risky security (a stock or a bond) which lasts for a single period. It can be traded (exchanged or borrowed/lent) in whatever quantity in the financial market at the beginning and at the end of each period. This security provides a positive expected return $E(R) > 0$ in all periods. Its return is revealed at the middle of each period. The security provides the following return at the end of each period t:

$$R_t = \begin{cases} R_h & \text{with probability } x \\ R_l & \text{with probability 1-}x \end{cases}$$

A.2. Depositors

Depositors are modelled as a continuum with population density equal to unity. In aggregate, depositors possess an equally divided endowment consisting of I units of cash. Despite being equally endowed with cash, depositors belong to two distinct kinds, which correspond to their distinct preferences: inpatient or patient. Inpatient depositors are affected by a liquidity shock at the middle of each period and need to have their cash back before the end of the period, in order to get utility out of it. Patient depositors, on the contrary, do not have this need and can wait until the end of the period to use their cash. Having defined $U(c_e)$ the depositors' utility deriving from the early withdrawal of cash and $U(c_l)$ the utility deriving from the late withdrawal, inpatient depositors have utility $U_i = U(c_e)$, while patient depositors have utility $U_p = U(c_l)$. The proportion of inpatient depositors is equal to λ. λ is a random variable that can take the following values in each period:

$$\lambda_t = \begin{cases} \lambda_h & \text{with probability } y \\ \lambda_l & \text{with probability 1-}y \end{cases}$$

While the probability distribution of λ is general knowledge, the specific value of the realisation of λ in each single period is unknown. At the middle of the period, each depositor discovers his own kind and acts accordingly. The kind each depositor belongs to is not observable.

Each depositor, at the beginning or each period, has the possibility to invests its entire amount of cash by entering a deposit contract with the dealer bank. According to the deposit contract, depositors lend their cash to the dealer bank at the beginning of the period. At the middle of the period, depositors can decide to roll on their deposit until the end of the period or to withdraw their deposit early. The possibility of an early withdrawal of the deposits accommodates the inpatient depositors need of an early use of cash. Because the kind of the depositors is not observable, the deposit contract has to eliminate any obstacle to the "kind revelation". For this reason, deposits withdrawn early (inpatient) have to be remunerated less than deposits withdrawn late (patient): $r_c < r_i < r_p$ (but more than cash). Having assumed that the depositors are able both to withdraw and redeposit at the middle of the period and that remuneration of the early withdrawn deposit is equal to the remuneration on the fresh cash deposited at the middle of the period, follows the following non-arbitrage condition for the remuneration of deposits:

Lemma 1:

$$1 + r_i = \sqrt{1 + r_p} \tag{1}$$

Proof. With a larger r_i patient depositors would be motivated to withdraw and redeposit immediately after, with a lower r_i early depositors would not withdraw and would prefer to sell their deposit to some other investor in exchange of cash. □

Finally, depositors are assumed to be risk neutral. Since we are analysing the funding of a dealer bank it is not necessary to postulate risk aversion in the depositors, which are, predominantly, non-retail investors.

A.3. Hedge fund

The HF is a risk neutral financial operator specialised in making arbitrage deals in the financial market. At the end of each period the HF has the opportunity to trade the security in the financial market at its fair value (the one revealed at the middle of the period). In addition, at the very beginning of each period the HF is offered the opportunity to trade the risky security, in whatever quantity, at a fixed price

$$P \neq E[R] \tag{2}$$

This price P can be either above or below the fair value of the security $E[R]$. This translates, for the HF, in the possibility of gaining an expected positive profit by buying/short-selling the security at the beginning of each period and selling/buying it back at the end of the period. Additionally, we assume no search or transaction costs for the HF trades in the financial market, meaning that an expected positive return is always available to the HF as long as (2) holds.

Please note that in the following we will analyse exclusively a first case in which $P < E[R]$. The reason for this is that arbitrage in a second alternative scenario where $P > E[R]$ would require short selling and we have assumed no endowment in securities for the HF. As it will be clear in the course of the analysis, assuming an endowment in securities S equal to $\frac{K}{E[R]}$ would make the second alternative scenario perfectly symmetrical to the first one. For the sake of clarity and simplicity we restrict the analysis to the first scenario.

The HF is endowed with own funds equal to K. K serves as the primary source of funding for its arbitrage trades. Given a market condition where $P < E[R]$, K can be used to buy $q = \frac{K}{P}$ units of the security (from now on just 'securities') at the beginning of the period and then wait the end of the period to sell them again in the financial market gaining an expected one-period profit of $E[\pi_{HF}] = K\frac{E[R_t]}{P}$. We assume that, in the case of a realised gain at the end of a period, the profits are completely distributed among the shareholders of the HF, while, in case of a realised loss, the shareholders immediately provide the necessary funds to bring K back to the original level. Which means assuming that the expected profits deriving from the HF business are bigger than the opportunity cost for the shareholders of providing the necessary economic capital to the HF.

Given the infinitely elastic supply curve faced by the HF in the market, expected profits are an increasing linear function of quantity of securities traded at the beginning of the period. For this reason, the HF has the incentive to tap the credit market to add financial leverage to its trading. HF can enter repo contracts with both DB and OFI at the beginning of the period. Repo contracts are settled at the beginning of the period and reversed at maturity, at the end of the period. In these repo contracts the HF uses the securities purchased in the financial market as collateral for borrowing cash from both DB and OFI. We assume that, in the bargaining between HF and OFI and DB, the HF is price taker: he is offered terms which he can accept (it enters the contract) or refuse (no borrowing takes place).

Though, the quantity of debt the HF can accumulate is limited. A solvency constraint, similar to the solvency ratios imposed by prudential regulations and in force in most jurisdictions, limits the increase of its financial leverage. The solvency constraint imposes that the worst case loss the HF can face has to be covered by the own available funds, that is to say:

$$q(P - R_l) - K \leq 0 \tag{3}$$

Assuming that the HF discounts future profits using a factor β and that the defined setting avoids any risk of interruption of the activity (such as a default), and indicating with r the interest rate paid to the creditors for the cash borrowed in the repo transactions, we can finally formulate the HF problem as follows:

$$\max_q E\left[\sum_{t=0}^{\infty} \beta^t (q(R_t - P) - r(Pq - K))\right] \tag{4}$$

subject to the constraint reported in (3).

The previous equations can be rewritten in terms of the HF total debt D_{HF}, employing the identities $D_{HF} = Pq - K$, $q = \frac{D_{HF}+K}{P}$ and $P = \frac{D_{HF}+K}{q}$:

$$\max_{D_{HF}} E\left[\sum_{t=0}^{\infty} \beta^t ((D_{HF} + K)\frac{R_t}{P} - D_{HF}(1 + r) - K)\right] \tag{5}$$

subject to the constraint:

$$D_{HF} - K\frac{R_l}{P - R_l} \leq 0 \tag{6}$$

In the HF problem as stated in (5) the only stochastic term is R_t. We can therefore rewrite (5) as:

$$\max_{D_{HF}} \left[\frac{1}{1-\beta}((D_{HF}+K)\frac{E[R]}{P} - D_{HF}(1+r) - K) \right] \tag{7}$$

which can be simplified to:

$$\max_{D_{HF}} \left[(D_{HF}+K)\frac{E[R]}{P} - D_{HF}(1+r) - K \right] \tag{8}$$

always subject to the solvency constraint (6).

A.4. Other financial intermediaries

We model OFI as a category of financial operators that makes business in close conjunction with the DB, but that can be also alternative to the DB. OFI could be money market funds, pension funds and other professional investors that make use of specific services provided by the bank, such as, for instance, the custody and trading of securities, the running of the checking account or the supply of consultancy services. In virtue of this close link, DB has a privileged access to funding from this category of operators. At the same time, OFI operate in the credit market in a bank-like manner (shadow banking). Being typically less leveraged than banks, OFI supply of funds to the credit market undergoes stricter feasibility conditions.

In the model, OFI can enter repo contracts with both the DB and the HF. As indicated in the section (II.A.3), we will restrict the analysis only to the case in which the HF arbitrages by buying securities in the financial market. For this reason the OFI will be asked by the HF to enter a repo contract in which the HF borrows cash, pledging the securities as collateral. In the opposite case, the OFI would have been asked by the HF to lend securities taking cash as collateral (securities lending). This last case can be modelled in exactly the same way as the first one, but for the sake of clarity and simplicity we restrict our analysis to the case in which OFI lend cash against securities.

We assume that, in the case OFI enter a repo contract with the DB, the conditions are more favourable: the interest rate charged to the DB is equal to the interest rate paid by the DB to the patient depositors (r_p). This is in line with the assumption that DB and OFI have a close and enduring business relationship that motivates favourable conditions. It is also consistent with the scale economies the DB can make in relating with the multitude of OFI, thanks to its being already in touch with the OFI due to other business relationships. OFI's supply curve of repo funds to the HF shows, on the contrary, a constant positive elasticity. The positive elasticity can be justified by the increasing cost of searching for creditors in a granular repo market. The supply curve is assumed to have the following functional form:

$$r = \phi D_{OFI} \tag{9}$$

where ϕ is the constant elasticity of the price (r) with respect to the quantity of debt (D_{OFI}) provided.

A.5. Dealer bank

The DB is the pivotal node of the model. It is modelled as a risk neutral deposit taking company extremely high leveraged, endowed with a negligible amount of own funds, which we will consider equal to 0 in the calculations. Deposits are gathered by the DB at the beginning of each period and invested in the only one profit generating activity available for the DB: the financing of the HF funding needs.

The DB can also enter repo contracts with the OFI, always with the aim of raising funds to be channelled towards the HF. The DB generates profits by providing intermediation between both depositors and OFI, on the one side, and the HF, on the other. The rationale for its existence is the reduction of the search and transaction costs ensured by its being a "one-stop shop" for all the other agents in the economy.

The DB activity is not completely exempt from risks. The main risk for the DB is a *liquidity risk*. Since the type of the depositors is not observable, all depositors can possibly ask to withdraw their deposits at the middle of the period. The DB is therefore exposed to the potential risk of runs by its depositors. In the described setting, though, depositors have no point in running the bank: as long as the HF solvency constraint (6) holds, any loss incurred by the HF will be covered by the own funds and the debt with the DB will be always repaid. The only constraint posed by the deposit taking activity is therefore the availability of the necessary funds to face the withdrawal requests by the *true* inpatient depositors at the middle of each period (liquidity constraint).

The liquidity constraint of the DB is strictly connected to the key policy parameter we investigate in this work: θ. θ represents the fraction of the collateral received by the HF that the DB can *re-use* in a further repo transaction with the OFI. Indicating with I the deposit gathered at the beginning of the period, the loan (D_{DB}) the DB will be able to extend to the HF will always respect the following *liquidity constraint*:

$$D_{DB} \leq \frac{I(1-\lambda_h(1+r_i))}{1-\theta} \tag{10}$$

where $I(1-\lambda_h(1+r_i))$ is equal to the maximum amount of deposits that can be used to finance activities extending beyond the middle of the period (what we can call the DB "long-term projects") and $\frac{1}{1-\theta}$ represents the maximum *collateral multiplier* implied by the policy choice of selecting a certain parameter θ. The rationale for the policy maker to regulate θ are the following: the bigger θ, the larger the amount of loans granted in the economy and the higher the aggregate level of leverage (the amount of capital, in our case K, is unchanged). Expressing this in a simple numerical example: with $I = 1$, $\lambda_h = 0.6$ and $r_i = 0.05$, in the case of $\theta = 0$ the maximum loan extendable will be equal to $D_{DB} = 0.37$, while in the case of $\theta = 0.5$ or 0.8 the maximum loan extendable will be equal to $D_{DB} = 0.74$ or 1.85 respectively.

Assuming that the DB discounts future profits with the same factor β as the HF, the DB problem can be summarised as follows:

$$\max_{D_{DB},r} E\left[\sum_{t=0}^{\infty}\beta^t(rD_{DB} - I\lambda_t r_i - I(1-\lambda_t)r_p - (D_{DB} - I(1-\lambda_h(1+r_i)))r_p)\right] \tag{11}$$

subject to the constraint reported in (10).

III. Equilibrium without runs

In this section, just like in the next one, we will analyse how the different variables can be expressed, in equilibrium, as a function of the policy parameter θ. We will analyse only the case in which the HF buys the security in the market and is willing to finance its long position borrowing cash from the DB and from the OFI (as already indicated in the sections II.A.3 and II.A.4). The second case, the one in which the HF short-sells the security in the financial market, pledging it as collateral in a repo transaction with both DB and OFI is completely symmetric to the first (under certain assumptions). For this reason, the conclusions obtained analysing the first also extend to the second.

In this section we will proceed as follows: first we will solve the HF problem as stated in (8) and (6); second we will *plug* the solution *into* the DB problem as stated in (11) and (10) in the form of additional constraints; we will then be able to find the equilibrium values for the main variables of interest (among others: the HF total debt D_{HF}, the DB loan to the HF D_{DB} and the equilibrium repo rate r) as a function of the policy parameter θ.

As discussed in II.A.3 the bargaining mechanism between HF and creditors at the beginning of the period renders the HF a *price taker*. His profit function and investment opportunities are assumed to be common knowledge. As will become clear in the next section, the HF's profits are strictly increasing in the amount of debt exposure, as long as the rate is lower than the marginal expected profitability of the arbitrage trade. Both the OFI and the DB know that, as long as the HF is offered a loan at a rate slightly lower than this threshold, the HF will maximise his profits by accepting the offered terms.

For the above mentioned reasons, as long as the maximum debt threshold is not hit (as long as the solvency constraint is not binding), the DB does not need to offer better conditions in order to gain market shares at the expense of the OFI. After the maximum debt threshold is reached, the DB can perfectly internalise in its problem the responses of both OFI and HF to a change in the rate: at a lower rate, the OFI will offer less credit to the HF and the DB will gain market shares accordingly. Being the profit of the HF monotonic increasing in the total debt and being the slope of the funds' supply curve of the OFI constant and of common knowledge, the DB is able to adjust the key variable, the equilibrium rate r, in order to maximise his own profits.

A. HF: equilibrium

In the HF problem as stated in (8), the maximisation of the single period expected profits provides the solution for the infinite periods problem. Indicating with ζ the Lagrange multiplier associated with the solvency constraint (6), the first order necessary condition for this problem and the complementary Kuhn-Tucker condition are the following:

$$\frac{E[R]}{P} - (1+r) - \zeta = 0 \tag{12}$$

$$\zeta \geq 0 \text{ with } \zeta = 0 \text{ if } D_{HF} - K\frac{R_l}{P - R_l} < 0 \tag{13}$$

These conditions involve that the HF profit cannot be maximised by any amount of debt, because it is a linear function of it. As long as the expected marginal return on the investment $\frac{E[R]}{P}$ is bigger than the marginal cost of the debt $1+r$, it will be always profitable for the HF to increase its debt. With $\frac{E[R]}{P} < 1+r$ HF enters no repo contract. With $\frac{E[R]}{P} = 1+r$ the HF is indifferent between entering and not entering a repo contract.

The solution to the HF problem is thus that the HF enters any repo deal in which the following two conditions are simultaneously satisfied: (i) the *solvency constraint* (6) and (ii) the following *participation constraint*:

$$r - \left(\frac{E[R]}{P} - 1\right) \leq 0 \tag{14}$$

B. *DB: equilibrium*

The equilibrium values for the interest rates remunerating the depositors have been partially dealt with in section II.A.2, in which we concluded with the non-arbitrage conditions reported in lemma (1). We can conclude the analysis of the funding conditions in the deposit market by formulating the following lemma:

Lemma 2: *Assuming the depositors discount future utility with the same factor β as the HF and the DB, there is only one set of equilibrium values for r_i and r_p, that makes possible the roll-over of the deposits across sub-periods according to the revealed kind of the depositor and make the DB shareholders willing to have the DB gathering deposits: $r_p = \frac{1}{\beta}$ and $1 + r_i = \sqrt{1 + r_p}$.*

Proof. Suppose that the rates were higher: the DB shareholder would prefer inject capital in the DB and let it finance via sole equity. If they were lower, no depositor would be willing to deposit its cash, since the present value of the deposit contract is negative.

In order to have deposits and credit market in place, the lemma 2 is a necessary but not a sufficient condition. Also the DB is subject to a participation constraint: in order to make it optimal for the bank to accept deposits and provide credit to the HF, the revenues deriving from this intermediation process have to cover, at least, the financing costs. Since the DB operates in an oligopoly regime, we assume the participation constraint is always satisfied for the DB and we do not take this constraint into consideration for the resolution of the DB problem and restrict our attention to areas of the parameters space where this constraint is not binding.

Having defined the equilibrium conditions for the DB on the funding side and having assumed in Section II.A.4 that the DB repo contracts *vis-a-vis* the OFI take place at a fixed predefined rate equal to the equilibrium funding rate r_p, we have all the necessary elements to reformulate the DB problem including the conditions derived from the HF equilibrium.

Starting from the problem enunciated in (11), firstly we note that also in the case of the DB problem, just as in the case of HF, the only stochastic element is λ_t. So the DB problem can be rewritten as a maximisation problem for the single period profit, which in turn is a function of $E[\lambda]$ (the t subscript is removed, since the analysis now regards a single period).

Secondly we can reformulate the problem in terms of D_{HF} and r instead of D_{DB} and r. This is possible, because of the particular relationship existing between the funding the HF receives from the OFI and from the DB. If the HF *solvency constraint* does not hold, it means that OFI and DB together are unable to finance the maximum debt the HF is willing to get. In this case it is pointless for the DB to select a rate lower than the maximum rate defined in the HF *participation constraint*, and the DB problem becomes degenerate and regards only how large is the loan the DB can extend. When the HF *solvency constraint* starts to be binding, the DB's decision on the quantity of loan to extend is no more trivial and the DB starts facing a trade-off: the increase in the quantity of loan extended will be accompanied by a decrease in the equilibrium interest rate. But given the rigid structure of the OFI funding and their predefined (and commonly known) supply curve, the DB can incorporate in its problem also the response to a lowering of the equilibrium rate of the OFI: they will simply reduce the amount of loans extended. More precisely, from the equation (9) and the structure of the market that makes $D_{HF} = D_{DB} + D_{OFI}$ true, the DB can anticipate that, assuming the HF *solvency constraint* is binding, the quantity D_{DB} will be just equal to:

$$D_{DB} = D_{HF} - \frac{r}{\phi} \tag{15}$$

which entails that lower the rate, the bigger the share of the D_{HF} that the DB is able to cover.

Thirdly, we simplify the notation summarising under the expression c_d (expected cost of deposits) and c_p (cost of the minimum level of patient deposits) the following quantities:

$$c_d = IE[\lambda]r_i + I(1 - E[\lambda])r_p \tag{16}$$

$$c_p = r_p(I(1 - \lambda_h(1 + r_i))) \tag{17}$$

Considering all above stated, the DB problem can be rewritten as follows:

$$\max_{D_{HF},r} \left[rD_{HF} - \frac{1}{\phi}r^2 - r_p D_{HF} + \frac{r_p}{\phi}r - c_d + c_p \right] \tag{18}$$

subject to three constraints:

$$D_{HF} - \frac{r}{\phi} - m_d \leq 0 \tag{19}$$

$$D_{HF} - m_k \leq 0 \tag{20}$$

$$r - m_r \leq 0 \tag{21}$$

where (19) is the DB *liquidity constraint*, (20) is the HF *solvency constraint* and (21) is the HF *participation constraint* and the expressions m_d, m_k, and m_r summarise:

$$m_d = \frac{I(1 - \lambda_h(1 + r_i))}{1 - \theta} \tag{22}$$

$$m_k = K \left(\frac{R_l}{P - R_l} \right) \tag{23}$$

$$m_r = \frac{E[R]}{P} - 1 \tag{24}$$

The Lagrange multipliers associated to (19), (20) and (21) are δ, ψ and γ respectively. The two necessary first order conditions and the three Kuhn-Tucker conditions for the restated DB problem are:

$$r - r_p - \delta - \psi = 0 \tag{25}$$

$$D_{HF} - \frac{2}{\phi} r + \frac{r_p}{\phi} + \frac{\delta}{\phi} - \gamma = 0 \tag{26}$$

$$\delta \geq 0 \text{ with } \delta = 0 \text{ if } D_{HF} - \frac{r}{\phi} - m_d < 0 \tag{27}$$

$$\psi \geq 0 \text{ with } \psi = 0 \text{ if } D_{HF} - m_k < 0 \tag{28}$$

$$\gamma \geq 0 \text{ with } \gamma = 0 \text{ if } r - m_r < 0 \tag{29}$$

The equilibrium solutions, found in Appendix A and expressed as functions of the policy parameter θ, identify two thresholds. The first one, that will be called *threshold 1*, identifies a first area of solutions, where only the HF *participation constraint* and the DB *liquidity constraint* are binding. The second, that will be called *threshold 2* delimits two areas: from *threshold 1* to *threshold 2* both DB *liquidity constraint* and HF *solvency constraint* are binding. Above *threshold 2* only the HF *solvency constraint* is binding.

$$\theta_1 = 1 - \frac{I(1 - \lambda_h(1 + r_i))}{m_k - \frac{m_r}{\phi}} \tag{30}$$

$$\theta_2 = 1 - \frac{2\phi I(1-\lambda_h(1+r_i))}{\phi m_k - r_p} \tag{31}$$

Below *threshold 1*, the equilibrium solutions for rate and HF debt are $r = m_r$ and $D_{HF} = \frac{m_r}{\phi} + m_d$. Between *threshold 1* and *threshold 2*, the equilibrium solutions are $r = \phi(m_k - m_d)$ and $D_{HF} = m_k$. Above *threshold 2* both equilibrium rate and total debt are fixed at the values $r = \frac{\phi m_k + r_p}{2}$ and $D_{HF} = m_k$ respectively.

The resolution of the stated problem, performed in the Appendix A, provides us with an interpretation of the equilibrium values as a function of the policy parameter θ leading us to the formulation of two proposition regarding the relationship between *re-use* of collateral and credit market.

Proposition 1: *Assuming the existence of a value of $\theta = \hat{\theta}$ that makes all the three constraints (19), (20) and (21) simultaneously binding (and this is a sensible assumption, since otherwise, the problem becomes trivial), as long as the HF solvency constraint is not the only one constraint binding, the HF as well as the overall (DB and HF) debt is an increasing function of θ. This means that the overall leverage is an increasing function of θ.*

Proof. Adopting as starting point $\hat{\theta}$, the reduction of θ, meaning that $\theta < \hat{\theta}$, can only make the DB *liquidity constraint* more stringent. Considering the OFI supply function and the fact that the HF *participation constraint* was holding in $\hat{\theta}$, it can only mean that a decrease in θ produces the HF *solvency constraint* not to hold anymore, which in turn means a reduction in the total HF debt D_{HF}. On the other hand, an increase in θ with respect to $\hat{\theta}$ cannot make the HF *solvency constraint* looser. The only two remaining possibilities are that the DB *liquidity constraint* ceases to hold or the same does the HF *participation constraint*. Since at $\hat{\theta}$ both rate and debt amount are constrained, debt can only be increased by letting the *participation constraint* cease to hold.

Proposition 2: *Under the same assumptions as in proposition (1), the equilibrium rate is a decreasing function of θ and the aggregate profits (DB and HF) in the market are an increasing function of θ. This means that the θ exerts an easing effect on credit markets.*

Proof. Always adopting as starting point $\hat{\theta}$, the reduction of θ makes the HF *solvency constraint* no more binding, meaning that less profits coming from the intermediation activity are generated, while the equilibrium rate is unchanged (the HF *participation constraint* still holds). Increasing θ, on the other hand, entails a decrease of the equilibrium rate (the HF *participation constraint* does not hold anymore) and an increase, for the same reason, of the HF profits. Also the DB increases its profits, until optimum.

C. *Numerical example*

Now we illustrate the results obtained analysing the *re-use* of collateral under no aggregate uncertainty with a numerical example. We assume a discount parameter β equal to 0.95 for all agents; a slope ϕ for the OFI funds supply curve equal to 0.3; market price P for the risky security, HF own funds K, and total deposits I all equal to 1; the proportion of inpatient depositors is assumed to take the values 0.6 and 0.4 with the same equal probability; the risky security is assumed to deliver a return with values 2.3 and 0.71 with the same equal probability ($E(R) = 1.505$).

For the definition of the HF problem, the only relevant parameters are the market price P for the risky security and HF own funds K and the expected return on the investment $\frac{E(R)}{P}$. Having set the first two parameters to 1 and the second to 1.505 deliver the equilibrium conditions represented in Figure 2. The relative profitability and the limited riskiness of the investment allow a leverage up to 2.44 (meaning with leverage the ratio between total debt and own funds). At a rate of 0.505 the HF becomes indifferent between borrowing and not borrowing cash (blue dotted line).

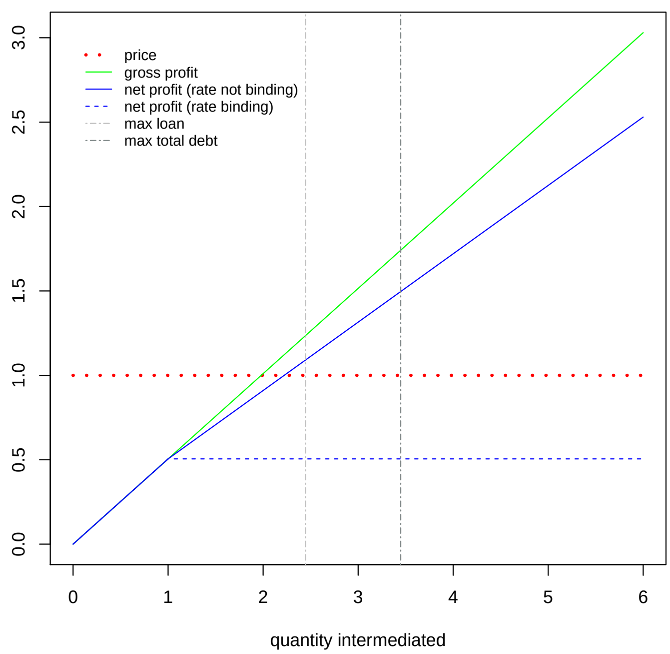

Figure 2. The HF problem

Given these parameters we can obtain the relevant cut-offs in terms of θ for our analysis. The first cut-off of interest is the one represented as a violet dotted line in Figure 3. It represents the value of θ for which all the three constraints in the DB problem are binding (in the example $\theta = 0.497$). The second cut-off corresponds to $\theta = 0.661$ and indicates the value of θ over which the DB *liquidity constraint* is no more binding. Figure 3 summarises how the equilibrium profit for the DB is determined by different constraints in different areas of the parameter θ space. Below the *threshold 1*, in the red area, the equilibrium rate is uniquely determined by the HF *participation constraint*. Loosening the DB *liquidity constraint* with an increase in θ leads to more DB loan and more profits. Between the *threshold 1* and the *threshold 2*, the HF *participation constraint* holds no more and the loosening of the DB *liquidity constraint* increases the loan volume, but at the expenses of a decreasing equilibrium rate. Above *threshold 2* any increase in the DB loan reduces the profit and the policy parameter θ does not influence profits any more.

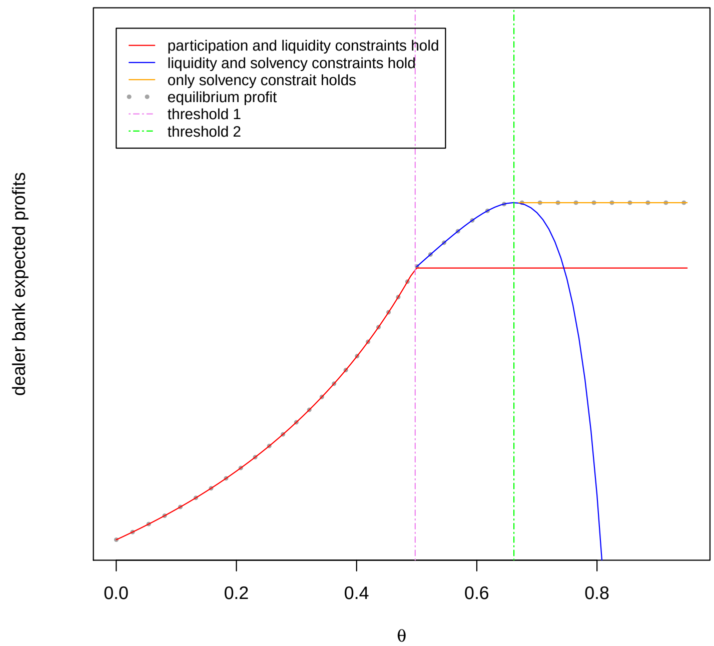

Figure 3. DB equilibrium profit levels

The capacity of the parameter θ to increase the aggregate level of debt and to ease the market conditions is illustrated in Figure 4. The plot A represents the equilibrium at a level of θ lower than *threshold 1*. In this area of the parameter θ values, the HF is unable to fully satisfy its demand for funds and a funds shortage takes place (the total debt is less than 2.2 against a demand for 2.44 units of debt). The plot B represents the equilibrium at a level of θ exactly equal to *threshold 1*. At this value of θ the HF demand of funds is fully satisfied, but at the highest cost (the rate implicit in the HF *participation constraint*). The plots C and D represent the equilibrium at a level of θ between *threshold 1* and *threshold 2* and exactly at *threshold 2*, respectively. In both cases the loan extended by the DB increases at the expenses of the OFI loans. The equilibrium rate decreases accordingly.

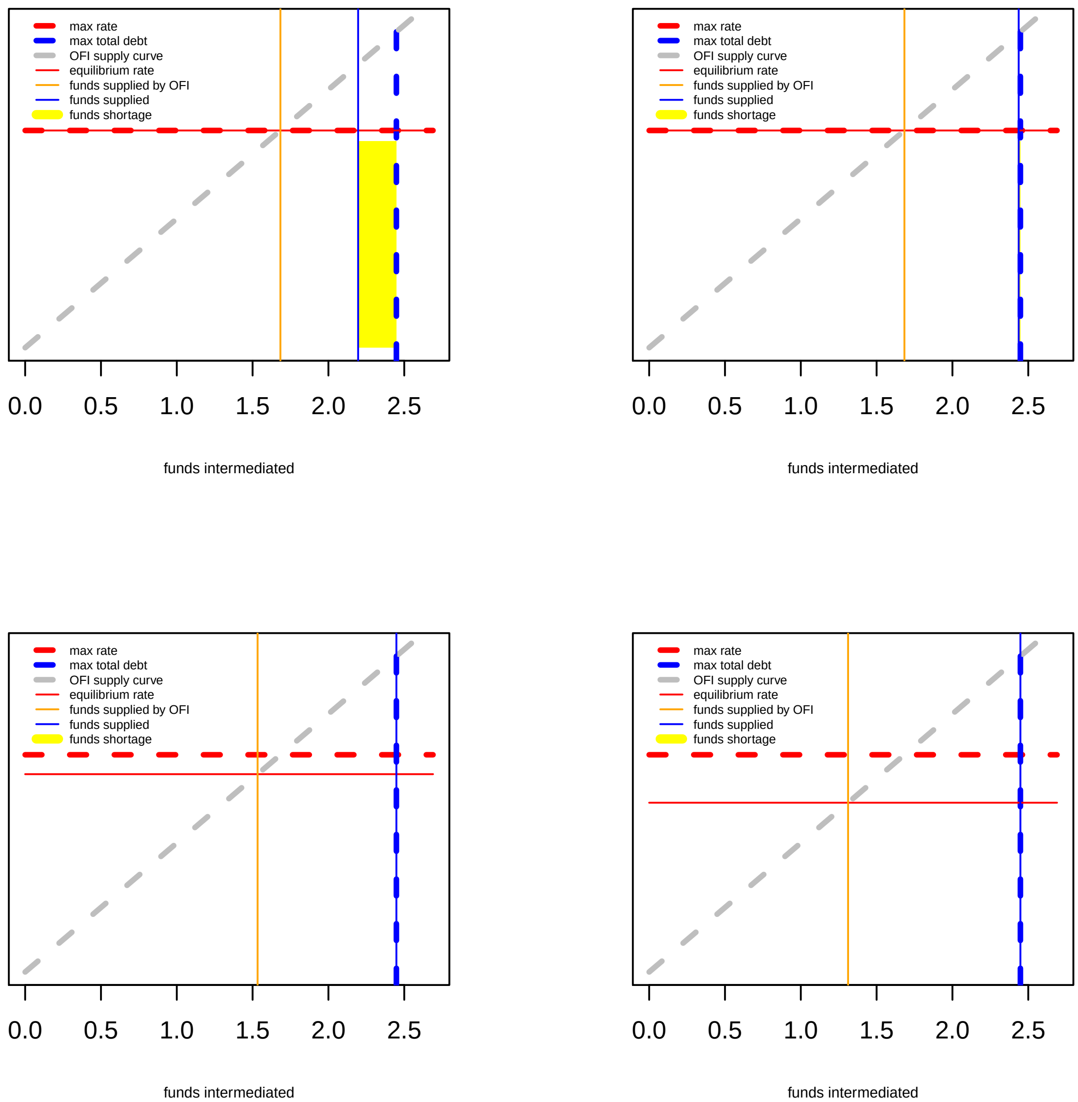

Figure 4. DB and OFI lending to the HF

Figures 5 and 6 represent the credit market conditions as functions of the policy parameter θ. While *threshold 1* is the point where the marginal profitability of new loans to the HF starts to decrease for the DB, is between *threshold 1* and *threshold 2* that the easing effect of an increase of θ is truly reflected in the market conditions. Between these two thresholds we see both a decrease in the equilibrium price and a corresponding increase in the HF profits. In the same span of θ values also the DB profits increase, highlighting an almost universal beneficial effect of the θ increase (OFI, on the contrary, decrease their profits).

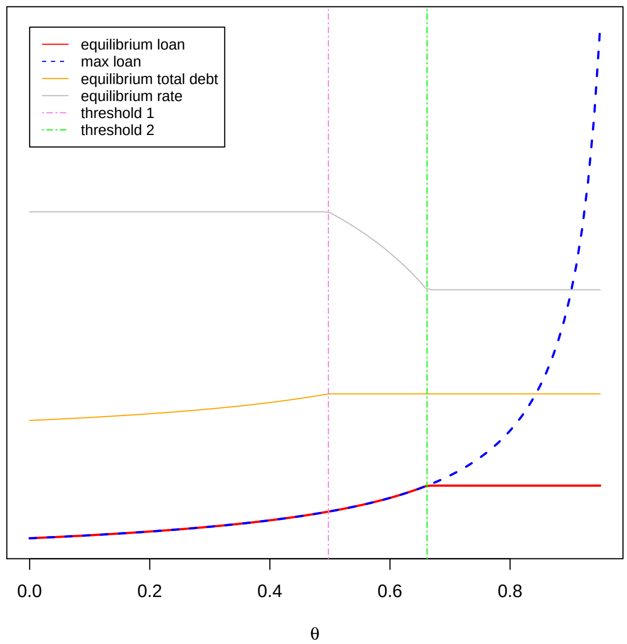

Figure 5. Credit market conditions

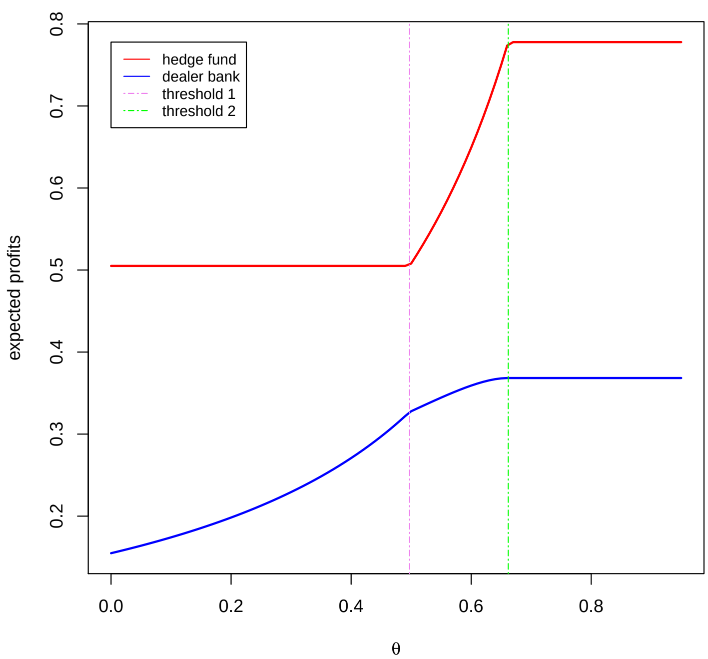

Figure 6. Market operators profits

The features of the above commented equilibrium values reflect on the value patterns of the Lagrange multipliers represented in 7. Below the *threshold 1* γ sees a sharp increase due to the increase in the shadow profit linked to the increased level of debt. δ is stable below *threshold 1* because the shadow profit linked to new debt is always equal to the maximum rate implicit in the HF *participation constraint.* Above *threshold 1* δ declines to zero as the marginal profitability of new debt approaches zero in *threshold 2.* ψ, on the contrary, increases between *threshold 1* and *threshold 2* because of the increasing marginal profit that would derive from lifting the HF *solvency constraint.* Above *threshold 2*, ψ remains stable since equilibrium values do not change anymore.

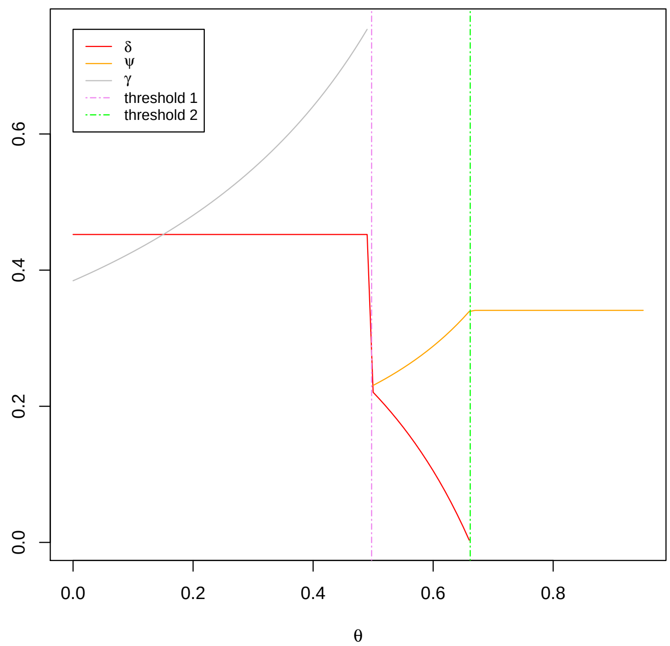

Figure 7. Lagrange multipliers analysis

IV. Stabilising and destabilising re-use

In the setting introduced in the previous section, the DB faced no significant uncertainty: the liquidity shock affecting the depositors at the middle of the period hardly affected the one period profits, did not affect the expected profits and, more importantly, was never able to trigger a bank run. On his part, the HF could incur losses, but an expected positive profitability of the HF business and a solid set of constraints was able to guarantee the solvency of the HF and to produce a rationale for the HF shareholders to recapitalise the HF, when needed, and prolong in this way the HF business indefinitely in the future.

This section introduces an additional element of uncertainty in the setting defined in the previous section. In particular, this section introduces the possibility for the HF to incur losses capable of making it close the business. These losses are triggered by the whole depreciation of the collateral, but must also be able to make the prosecution of the activity impossible for the HF. This kind of losses can be thought of as a tail loss, not offset by the solvency constraint, which, also in the reality, never provides coverage for the 100 per cent of the losses. For instance, they could be losses related to the trading activity, such as the security value going unexpectedly to 0. The trading losses could be additionally linked to reputational damage or to lawsuits and legal costs, maybe also linked to the consequences of fraud: all this producing a scenario where the prosecution of the business is no longer possible. This risk is common knowledge in the economy and all the agents are aware of the likelihood of this event, let it be named p_{def}, which takes the following values:

$$p_{def} = \begin{cases} p, & \text{for } D_{HF} \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The default of the HF becomes common knowledge at the middle of the period, just before the depositors undergo the liquidity shock, discover their type and can withdraw their bank account from the DB. The default of the HF propagates though the credit system via the *repo chains*. Simultaneously, the HF becomes insolvent and the collateral has no more value. The DB incurs losses in two ways: being a collateral holder (a fraction $1 - \theta$ of the collateral is in its portfolio), faces the need of recapitalisation to repay its patient depositors. Being in a repo contract with the OFI, the choice of not reversing the repo at period's end will make future business with the OFI impossible (this can be thought of as a consequence of retaliation or of an induced OFI crisis). For these reasons, the DB has two choices: either to make up for the entire amount of losses (the entire amount of the DB loan) or to close the activity in his turn.

It is common knowledge that the DB, in case of HF default, is faced with two alternative options: the default or the prosecution of its activity. As a consequence, the DB is now faced with the risk of a run: if patient depositors assess that the DB will go down the path of a termination of the business, they will no more wait until the period's end to withdraw their cash, but they will run the bank together with the inpatient ones in order to have a share of the DB available assets (the liquidity set aside to face $\lambda_h I$ withdrawals). More precisely, in the wake of a HF default, depositors are able to evaluate whether it is profitable or not for the DB shareholders to recapitalise or not. This entails some opaqueness in the DB business: as long as no default takes place, we assume the depositors are unable to look into the DB strategy and understand whether it will lead to a *domino* effect or not. But this veil is lifted as soon as the HF defaults (for instance, as the effect of a wake-up call)s and the DB strategy can be then correctly assessed. As a consequence, it is assumed that the remuneration of deposits is not affected by the presence of the additional risk of a domino default chain affecting the DB.

In the following sub-section the previous HF problem will be reformulated, taking into account the new default risk introduced in the model. The reformulated HF problem will provide a revised *participation constraint* for the HF that will be *plugged into* the DB problem, just like in the previous section, to solve the model and find the equilibrium values of the variables of interest.

As usual, the equilibrium values will be expressed as functions of the policy parameter θ and the effects of θ on the stability of the financial system will be assessed.

A. HF: equilibrium with uncertainty

In the new setting, the HF faces a fixed cost linked to the access to external finance: the cost consists in the possibility of the early termination of the business activity and the connected loss of the earnings flow following the default period. In the case the HF has no access to external finance (no debt), the equilibrium conditions are exactly the same as in the Section III (but with no external finance, so no repo market is there anymore). In the case the HF has some debt, then the duration x of its activity becomes a random variable with a geometric distribution where each possible value x has probability:

$$Prob(duration = x) = (1-p)^x p \tag{32}$$

Having expressed the sum of the first x elements of the geometric series with ratio β as $\beta^0 + \beta^1 + \beta^2 + ... + \beta^{x-1} = \sum_{i=0}^{x-1} \beta^i = \frac{1-\beta^x}{1-\beta}$, the problem of the HF can be rewritten as the maximisation of the expected value of the discounted profits that the HF can make over a random number of periods x:

$$\max_{D_{HF}} \left[\sum_{x=0}^{\infty} \frac{1-\beta^x}{1-\beta} \left((D_{HF}+K)\frac{E(R_t)}{P} - D_{HF}(1+r) - K \right) (1-p)^x p \right] \tag{33}$$

Also in this case, as in the case without probability of default, the only one stochastic element in the formula is R_t. On the other hand we can rely on the two following identities and summarise the summations of the discount terms (in both cases, the risky and the safe one - the one without outstanding debt and consequently no risk of default):

$$\omega_r = \frac{1-p}{1-\beta(1-p)} = \sum_{x=0}^{\infty} \frac{1-\beta^x}{1-\beta}(1-p)^x p \tag{34}$$

$$\omega_s = \frac{1}{1-\beta} = \sum_{t=0}^{\infty} \beta^t \tag{35}$$

Having made the appropriate substitutions the problem of the HF appears only slightly modified with respect to the case of no default risk. The probability of default is irrelevant for the determination of the HF equilibrium:

$$\max_{D_{HF}} \left[(D_{HF}+K)\frac{E[R]}{P} - D_{HF}(1+r) - K \right] \omega_r \tag{36}$$

What really changes is the HF *participation constraint*. In order to be profitable for the HF to have outstanding debt, the flow of expected profits deriving from this first strategy has to be larger than the flow of expected profits in the case of no outstanding debt:

$$\left[(D_{HF}+K)\frac{E[R]}{P}-D_{HF}(1+r)-K\right]\omega_r \geq \left(\frac{E[R]}{P}-1\right)K\omega_s \tag{37}$$

Which can be rewritten as:

$$D_{HF}\left(\frac{E[R]}{P}-(1+r)\right) \geq K\left(\frac{E[R]}{P}-1\right)\left(\frac{\omega_s}{\omega_r}-1\right) \tag{38}$$

The new HF *participation constraint* can be interpreted as follows: the marginal profitability of debt (the left-hand side term) has to be bigger than the marginal profitability of own capital multiplied by a coefficient that "rewards" the safety of the "no leverage" strategy (the right-hand side term). Summarising the right-hand side term in equation (38) in the expression m_p, we can finally conclude that the two conditions passed to the DB for the determination of the equilibrium are the usual HF *solvency constraint* (39) and a modified HF *participation constraint* (40):

$$D_{HF}-K\frac{R_l}{P-R_l} \leq 0 \tag{39}$$

$$r-\frac{E[R]}{P}+1+\frac{m_p}{D_{HF}} \leq 0 \tag{40}$$

Interestingly, in the new setting, the redefined HF *participation constraint* (40) is less and less binding the more D_{HF} grows. The non-linear relationship between boundary rate and debt makes it possible that a softening of the liquidity constraint on the side of the DB is more than linearly rewarded in terms of expected profits.

B. *DB: equilibrium with uncertainty*

For the sake of clarity and simplicity, in the following three crucial assumptions will be assumed as valid:

Assumption 1: *The elasticity and the behaviour of the OFI, as well as the remuneration of the depositors remain unchanged in the presence of a risk of default on the HF side.*

Assumption 2: *In case of HF default, the DB has two options: either (i) terminate the activity, or (ii) inject a quantity D_{DB} of fresh capital to repay creditors (depositors and OFI).*

Assumption 3: *In case of HF default, the next period a new HF enters the market and substitutes the defaulted HF.*

Given these assumptions, the DB can choose long before the HF default his own strategy. In a first strategy, that can be defined the "risky" strategy, the DB decides to run the business until the HF defaults and then to terminate it. In this case the DB acts according to the same criteria defined in the Section III and produces the same equilibrium results. In a second strategy, that can be defined the "safe" strategy, the DB decides to make the business sustainable also in the face of the HF's default. In this case the DB *tunes* the quantity of loan extended to the HF taking into account the possible losses entailed by the intermediation activity. What makes the DB choose between one of the two possible strategies is the present value of the earnings flow deriving from the two strategies. Using a simplified notation, the problem of the DB in case of HF default risk becomes a choice between the following two expectations:

$$E[\pi_{\text{risky strategy}}] = E[\pi_t]\frac{1-p}{1-\beta(1-p)} \tag{41}$$

$$E[\pi_{\text{safe strategy}}] = E[\pi_t - D_{DB}p]\frac{1}{1-\beta} \tag{42}$$

In what follows, the equilibrium values for the two strategies will be derived. General conclusions regarding the impact of the parameter θ on the final equilibrium values will be formulated in two specific propositions. The analysis of the relationship between the values of θ and the dominance of one strategy over the other will be conducted analytically and exploring the model parameters' space in the numerical example.

B.1. DB: equilibrium with uncertainty - risky strategy

As stated in the equation (41), the DB problem with HF default risk is just the regular DB problem analysed in the Section III with a modification in the HF *participation constraint*. Analytically:

$$\max_{D_{HF},r}\left[rD_{HF} - \frac{1}{\phi}r^2 - r_pD_{HF} + \frac{r_p}{\phi}r - c_d + c_p\right] \tag{43}$$

subject to three constraints:

$$D_{HF} - \frac{r}{\phi} - m_d \leq 0 \tag{44}$$

$$D_{HF} - m_k \leq 0 \tag{45}$$

$$r - \frac{E[R]}{P} + 1 + \frac{m_p}{D_{HF}} \leq 0 \tag{46}$$

where (44) is the DB *liquidity constraint*, (45) is the HF *solvency constraint* and (46) is the HF "new" *participation constraint*.

The Lagrange multipliers associated to (44), (45) and (46) are δ, ψ and γ respectively. The two necessary first order conditions and the three Kuhn-Tucker conditions for the new DB problem are:

$$r - r_p - \delta + \frac{\gamma m_p}{D_{HF}^2} - \psi = 0 \tag{47}$$

$$D_{HF} - \frac{2}{\phi}r + \frac{r_p}{\phi} + \frac{\delta}{\phi} - \gamma = 0 \tag{48}$$

$$\delta \geq 0 \text{ with } \delta = 0 \text{ if } D_{HF} - \frac{r}{\phi} - m_d < 0 \tag{49}$$

$$\psi \geq 0 \text{ with } \psi = 0 \text{ if } D_{HF} - m_k < 0 \tag{50}$$

$$\gamma \geq 0 \text{ with } \gamma = 0 \text{ if } r - \frac{E[R]}{P} + 1 + \frac{m_p}{D_{HF}} < 0 \tag{51}$$

The derivation of the equilibrium solutions is left to the Appendix B. The equilibrium solutions expressed as a function of the policy parameter θ allow to identify two thresholds (one of which, the second, is precisely equal to the one defined for the setting without HF default risk):

$$\theta_1 = \frac{I(1 - \lambda_h(1 + r_i))}{\frac{1}{\phi}\left(\frac{E[R]}{P} - 1 - \frac{m_p}{m_k}\right) - m_k} + 1 \tag{52}$$

$$\theta_2 = 1 - \frac{2\phi I(1 - \lambda_h(1 + r_i))}{\phi m_k - r_p} \tag{53}$$

Below *threshold 1*, the equilibrium r and D_{HF} are different with respect to the default risk-free scenario, this is due to the new HF *participation constraint* that forces the DB to adjust the rate for each specific quantity of loan extended. On the contrary, above *threshold 1*, nothing is changed with respect to the previous scenario: between *threshold 1* and *threshold 2*, the equilibrium $r = \phi(m_k - m_d)$ and the equilibrium $D_{HF} = m_k$; above *threshold 2* both equilibrium rate and total debt are fixed at the values $r = \frac{\phi m_k + r_p}{2}$ and $D_{HF} = m_k$ respectively. This leads to a first intermediate result summarised by the following lemma.

Lemma 3: *The presence of the a HF default risk does not modify the equilibrium conditions of the DB problem, in the case the "risky" strategy is selected, when we consider the area above threshold 1.*

Proof. In the DB problem with HF default risk, in case the "risky" strategy is selected, the only introduced alteration of the original problem is the HF "new" *participation constraint.* But this constraint stops being binding exactly where it is expected that the paths of the "risky" and of the "safe" strategy depart from each other: when the trade-off between increase in loan volume and maintenance of a profitable equilibrium rate materialises (after the *threshold 1*).

B.2. DB: equilibrium with uncertainty - safe strategy

According to equation (42), the DB problem with HF default risk, in case the "safe" strategy is selected appears essentially different from before: the presence of a default risk and the necessity to make provisions for it (the term $-D_{DB}p$ essentially expresses that the profit has to be accounted for net of the provisions on expected loan losses) introduces an additional cost in the DB profit function that is exclusively related to the loan quantity and not to the rate (the marginal profitability of the loan itself). Analytically the problem is the following:

$$\max_{D_{HF},r}\left[rD_{HF}-\frac{1}{\phi}r^2-r_pD_{HF}+\frac{r_p}{\phi}r-c_d+c_p-pD_{HF}+p\frac{r}{\phi}\right] \tag{54}$$

subject to the three constraints already reported in equations (44) - the DB *liquidity constraint*, (45) - the HF *solvency constraint* and (46) - the HF "new" *participation constraint*.

The Lagrange multipliers associated to (44), (45) and (46) are always δ, ψ and γ respectively. The two necessary first order conditions and the three Kuhn-Tucker conditions for the new DB problem are:

$$r-r_p-p-\delta+\frac{\gamma m_p}{D_{HF}^2}-\psi=0 \tag{55}$$

$$D_{HF}-\frac{2}{\phi}r+\frac{r_p}{\phi}+\frac{p}{\phi}+\frac{\delta}{\phi}-\gamma=0 \tag{56}$$

$$\delta\geq 0 \text{ with } \delta=0 \text{ if } D_{HF}-\frac{r}{\phi}-m_d<0 \tag{57}$$

$$\psi\geq 0 \text{ with } \psi=0 \text{ if } D_{HF}-m_k<0 \tag{58}$$

$$\gamma\geq 0 \text{ with } \gamma=0 \text{ if } r-\frac{E[R]}{P}+1+\frac{m_p}{D_{HF}}<0 \tag{59}$$

The derivation of the equilibrium solutions is left to the Appendix C. The equilibrium solutions as functions of the policy parameter θ identify two thresholds. The first one is the usual *threshold 1* as in the previous problem. The second, that will be called *threshold 3* is new and identifies the point in which the DB *liquidity constraint* is zeroed:

$$\theta_1=\frac{I(1-\lambda_h(1+r_i))}{\frac{1}{\phi}\left(\frac{E[R]}{P}-1-\frac{m_p}{m_k}\right)-m_k}+1 \tag{60}$$

$$\theta_3=\frac{I(1-\lambda_h(1+r_i))}{\frac{1}{2\phi}\left(r_p+p-\phi m_k\right)}+1 \tag{61}$$

Below *threshold 3*, the equilibrium r and D_{HF} are the same as in the "risky" strategy scenario. On the contrary, above *threshold 3* the DB stops, in the "safe" strategy scenario, to provide credit *earlier on* than in the "risky" strategy scenario. This is consistent with the intuition that for the "safe" strategy to work, it is necessary for the DB that the additional income brought by the new loan extended is at least equal to the corresponding necessary provisions to be made for the corresponding expected losses. This leads to a first intermediate result, summarised by the following lemma.

Lemma 4: *The increase in the parameter θ produces a differentiation between the equilibrium solutions of the "risky" and of the "safe" strategy only if the HF participation constraint is no more binding.*

Proof. In the case that the HF *participation constraint* is binding the two strategies cannot differentiate each other. The structure of the HF credit demand curve renders any additional unit of loan more profitable than those preceding. On the contrary, the provisions corresponding to each new unit of loan are constant. For this reason, the first derivative of the profit function in that area cannot converge to the indifference point $\pi^{'}(D_{DB}) = p$.

B.3. DB: equilibrium with uncertainty - conclusions

The derivation of the equilibrium results for the DB problem with HF default risk, provided in what precedes, preludes to the determination of what strategy dominates the other for the different values of the policy parameter θ and of the other model's parameters. For the "safe" strategy to be dominant for the DB, the following inequality must be satisfied:

$$E[\pi_t - D_{DB}p]\frac{1}{1-\beta} > E[\pi_t]\frac{1-p}{1-\beta(1-p)} \tag{62}$$

with $E[\pi_t]$ always denoting the one period profit generated by the DB activity and $D_{DB}p$ being the associated cost in terms of provisions on expected losses. Being $D_{DB}p$, once D_{DB} is chosen, certain, the inequality can be reformulated as:

$$E[\pi_t] > D_{DB}(1-\beta(1-p)) \tag{63}$$

That is to say that the one period profit has to be bigger than the opportunity cost for the DB, corrected by the HF default risk. The inequality clearly identified the relationship between dominant strategy and β and p: while a higher β is beneficial for the financial stability (it discourages the risk-prone and short-sighted "risky" strategy), conversely a higher p is detrimental for the stability of the market. But how does θ influence the DB strategy? θ influences the DB choices by means of its relationship with $E[\pi_t]$. On the one hand an increase in θ increases the one period profits of the DB. By means of this channel, an increase in θ strengthens the financial stability. On the other hand, the maximum leverage reachable under the "safe" strategy is lower than the one reachable adopting the "risky" strategy (*threshold 3* $<$ *threshold 2*). For this reason, an increase in θ can also exert a negative effect on financial stability: beyond *threshold 3* all increases in θ can only risk to make the "risky" strategy take the upper hand.

The main result of the analysis is the identification of these two channels: the *destabilising* channel of θ, by means of which θ increases the profits in areas of the parameters where the "safe" strategy cannot increase its leverage, making the "risky" strategy preferable. And the *stabilising* channel of θ, by means of which it increases the DB profits in areas of the parameters where increases in leverage are still sustainable, making the "safe" strategy preferable. We summarise the findings in the following two propositions.

Proposition 3: *For non-trivial portions of the model parameters' space, an increase in θ destabilises the financial system, determining an increase in the gains deriving from the leveraged lending business and, subsequently, an increase in the probability that the DB selected strategy is of the type "risky".*

Proof. The main channel by means of which θ is able to destabilise the financial system is the creation of an incentive for the DB to pursue a "risky" strategy. A dominant "risky" strategy is possible, because there is a set of non-trivial combinations of parameters that delivers a larger flow of discounted profits to the bank pursuing the "risky" strategy. In that case θ destabilised the financial system by introducing a perfect correlation between HF default and DB default, which means a certainty of *contagion* across financial institutions.

Proposition 4: *For non-trivial portions of the model parameters' space, an increase in θ stabilises the financial system, determining an increase in the profitability of the intermediation activity and rendering the DB "safe" strategy preferable to the "risky" one.*

Proof. The main channel by means of which θ is able to stabilise the financial system is the boosting of gross profits for the DB. In presence of a very low profitability of the DB activity, the "risky" strategy can be determined by an incentive for "gambling for resurrection". The increase in profits generated by higher θ values can in this case serve as a remedy for a financial system affected by low profitability issues.

C. Numerical example

The results obtained analysing the *re-use* of collateral in the scenario of HF default risk are now illustrated with two numerical example. The first one investigates the equilibrium values using a set of parameters values that leads to destabilising collateral *re-use*. The following values are assumed: a discount parameter β equal to 0.84 for all agents; a slope ϕ for the OFI funds supply curve equal to 0.32; market price P for the risky security, HF own funds K, and total deposits I all equal to 1; the proportion of inpatient depositors is assumed to take the values 0.6 and 0.4 with the same equal probability; the risky security is assumed to deliver a return with value 2.6 and 0.7 with the same equal probability ($E(R) = 1.535$). The probability of a HF default is set to 0.07.

Also in this case, for the definition of the HF problem, the only relevant parameters are the market price P for the risky security, the HF own funds K and the expected return on the investment $\frac{E(R)}{P}$. In addition, this time, the HF has to decide between having and not having debt and this decision requires also the parameters β and p. Having set the first two parameters to 1 and the others as above stated delivers the equilibrium conditions represented in Figure 8. The relative profitability and the limited riskiness of the investment allow for a leverage up to 2.03 (always meaning with leverage the ratio between total debt and own funds). This time the HF "demand curve" is parabolic and strictly increasing, mirroring the behaviour of the HF *participation constraint*. In addition, the plot reports the level of the DB *participation constraint*: operating in oligopoly regime, the DB can make sure always to select equilibrium values for r closer tot the red rather than to the violet line, ensuring its own *participation constraint* is not binding.

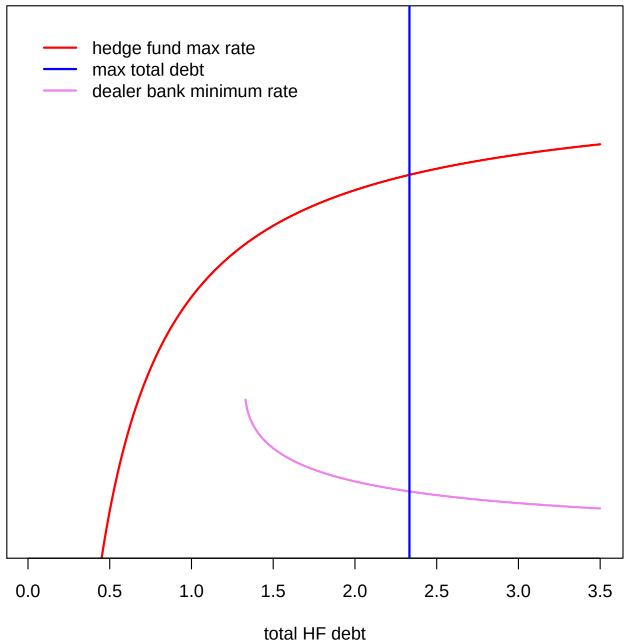

Figure 8. HF problem with default risk

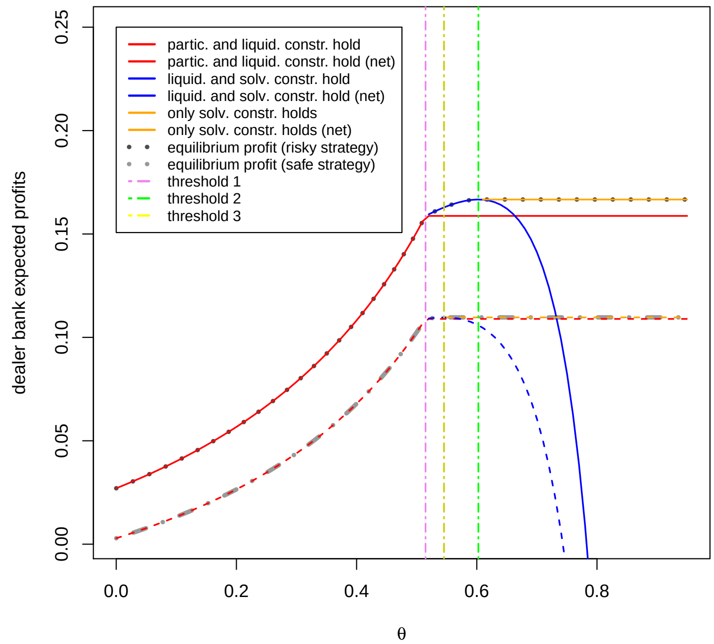

Figure 9. DB equilibrium profits with HF default risk

Figure 9 clearly shows the modifications in the equilibrium values determined by the new setting. The single-period equilibrium profits of the DB are highlighted for both the "risky" and the "safe" strategy. Notably the *threshold 3* is hit by the "safe" strategy before the "risky" strategy hits the *threshold 2*. The effect is a larger amount of loans granted in the "risky" strategy. The equilibrium profit line in the "safe" strategy lies below the "risky" strategy equivalent because of the impact of the "provisions", the loan loss the DB has to yearly detract from his gross expected profits if it wants to select equilibrium values compatible with the repayment of the losses in case of HF default.

The effects in terms of market conditions determined by the two competing strategies is represented in Figure 10. As anticipated from the previous analysis, the "safe" strategy limits the "easing" effects of the increases in the policy parameter θ. In case of a dominant "safe" strategy, the equilibrium rate is strictly higher and the equilibrium loan provided by the DB strictly lower. That is to say: lower leverage and lower liquidity in the marketing.

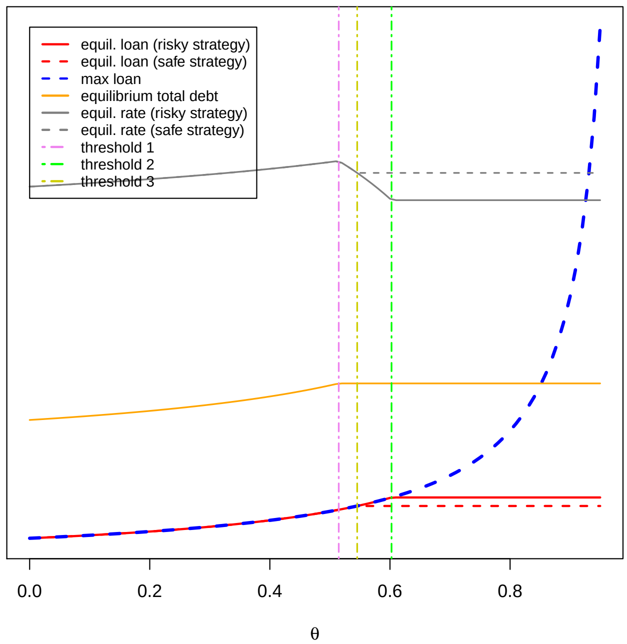

Figure 10. Market conditions with HF default risk

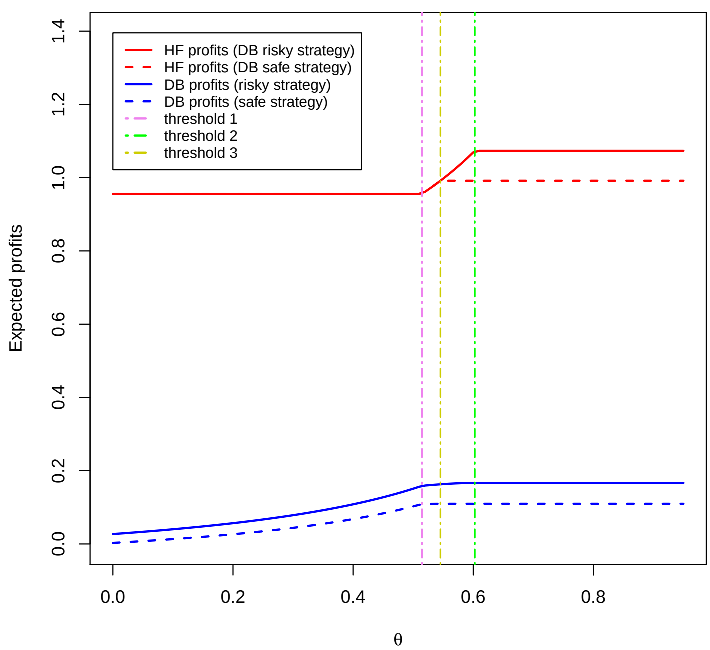

Figure 11. HF and DB profits with HF default risk

The change in the equilibrium market conditions reflects on the overall profits generated by the two financial operators. In both the case of the HF and of the DB the "safe" strategy depresses the profits, despite the beneficial effect of the increase in the policy parameter θ.

Lower equilibrium DB loans and profits deriving from the "safe" strategy make it possible for the "risky" strategy to become dominant just as a consequence of the increase in the policy parameter θ. This is made clear by the Figure 12. Approaching *threshold 3*, the "safe" DB one-period profits stabilise, bringing the present value of the DB to its final equilibrium level. On the contrary, the increase in θ allow the "risky" DB one-period profits to grow further, making the "risky" strategy dominant despite a substantial premium for the safety ($\omega_r = 4.25 < \omega_s = 6.25$). Interestingly, in this example, θ works as destabilising in both extremes: when extremely low and extremely high. In such a setting, the optimal choice of θ for the benefit of financial stability is identified in the middle range values.

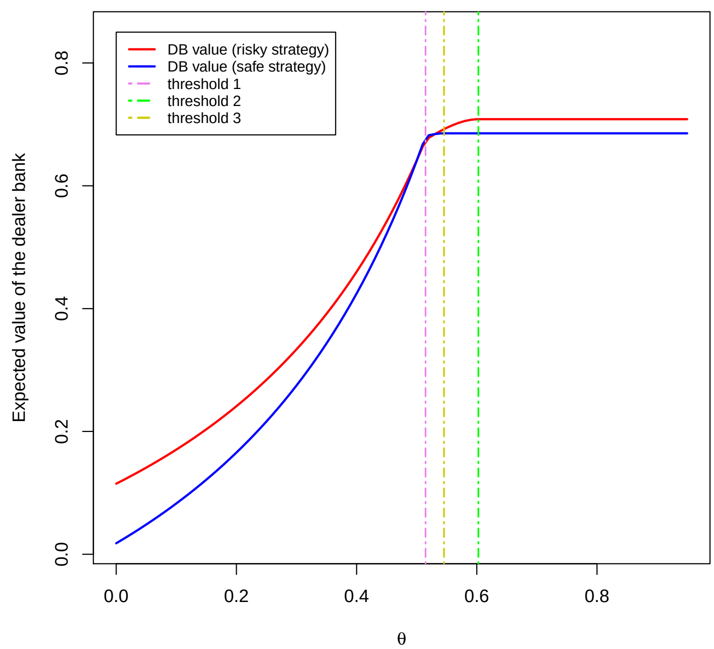

Figure 12. Present value of the DB with HF default risk

The situation is reversed in the second numerical example. Here the parameters used are slightly different: a discount parameter β equal to 0.85 for all agents; a slope ϕ for the OFI funds supply curve equal to 0.3; market price P for the risky security, HF own funds K, and total deposits I all equal to 1; the proportion of inpatient depositors is assumed to take the values 0.6 and 0.4 with the same equal probability; the risky security is assumed to deliver a return with value 2.3 and 0.7 with the same equal probability ($E(R) = 1.65$). The probability of a HF default is set to 0.02.

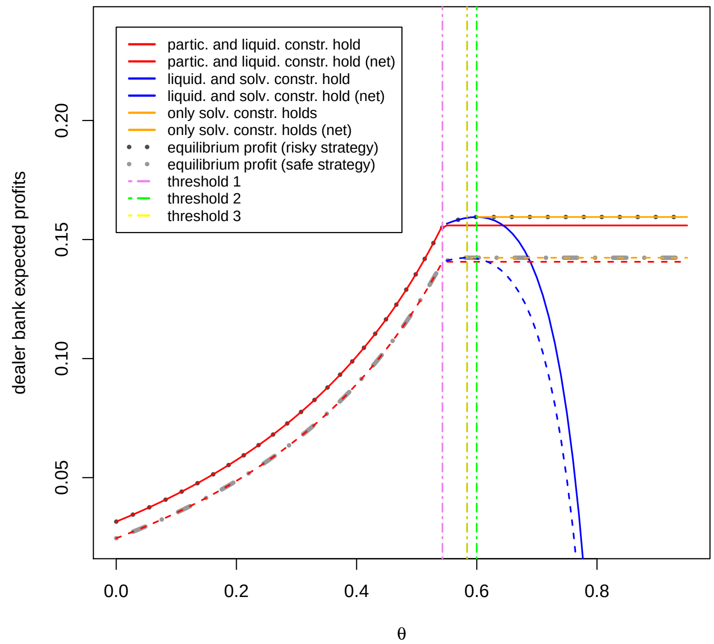

Figure 13. DB equilibrium profits with HF default risk (stabilising)

As shown in Figure 14, under the new parameters the increase in the policy parameter θ can exert a beneficial effect on the financial stability of the economy exactly for the reason identified in Proposition 4. By increasing the volume of lending allowed, the "safe" DB can slowly climb out of its profitability low and reach a dominant level of stable profitability as illustrated in Figure 14 and Figure 15.

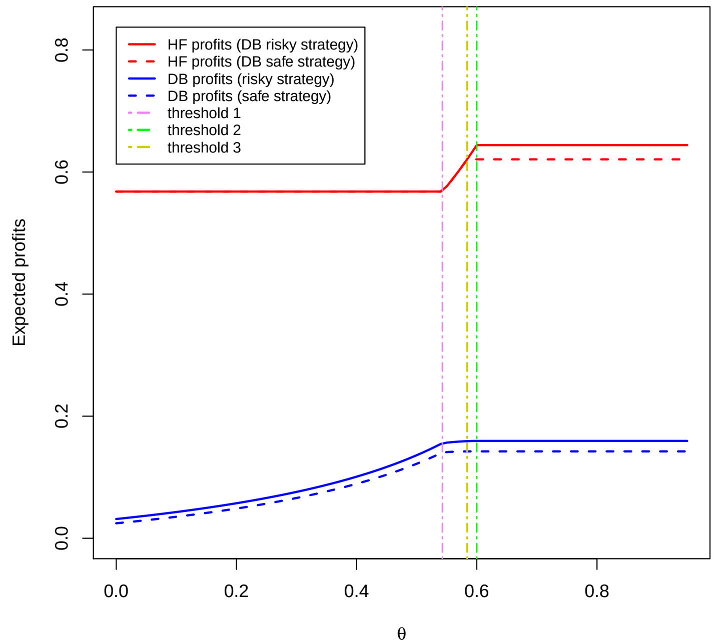

Figure 14. HF and DB profits with HF default risk (stabilising)

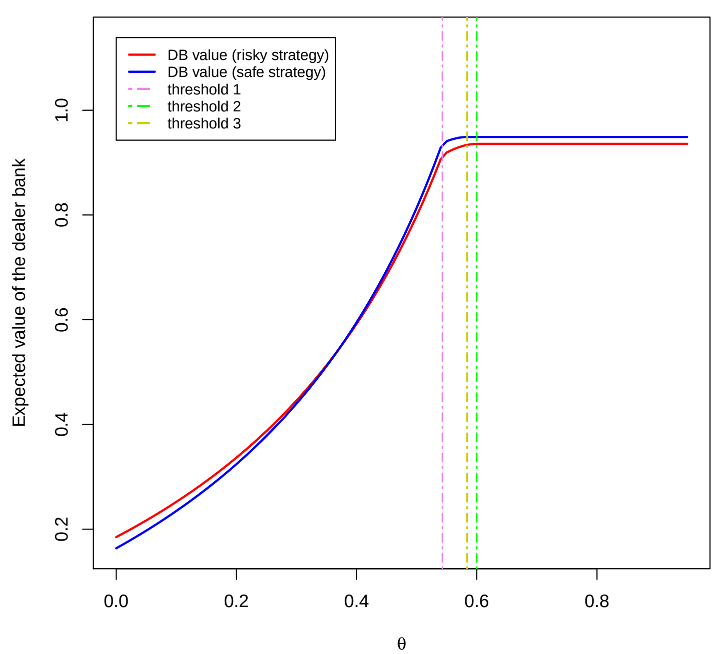

Figure 15. Present value of the DB with HF default risk (stabilising)

V. Discussion

Both the rationale and the consequences of the re-use of collateral have already been investigated in other studies. As already recalled in the introduction to this work, the literature has particularly focussed on the beneficial effects on the overall welfare like in Andolfatto et al. (2017). These beneficial effects are typically mediated by an increase in liquidity and by overall better market conditions, like those endogenously originating in the model by Gottardi et. al. (2017), where *repo chains* naturally originate once established the correct framework for the repo contracts. Where the negative spillovers of *repo chains* and *collateral re-use* on financial stability are modelled, it typically happens via the modelling of *fire sales*, like in Antinolfi et al. (2015) or *dry-ups*, reminiscent of the trade-off identified by the seminal paper of Brunnermeier and Pedersen (2009), or also volatility and inequality, like in Brumm (2018).

The present work endeavours to analyse the relationship between *collateral re-use* and financial stability from an original perspective, where the *collateral re-use* can both exert positive and negative effects, depending on the profitability of the alternative business models potentially at work in the market. In order to analyse in an encompassing framework both the mechanics of the repo market and the overall business strategy of the dealer bank involved, a relatively wide set of agents has been introduced into the model, while simplifying on the treatment of some mechanisms.

On the one hand, it has been possible to model with a relatively good level of detail the funding structure of the dealer bank, including in the picture both unsecured "demand" deposits and wholesale secured funding. The possibility of runs by depositors is particularly important to characterise the fragility of the repo market in terms of correlation (in the reality, higher or lower, in the model 0 or 1) between HF default (that in the reality could be any shock on the side of the DB professional clients caused by an unexpected move in the traded securities) and DB default (in the reality it could also be a downgrade, or the need for state help).

On the other hand, the liquidity provision of the repo market has been portrayed in a highly simplified manner. Though, the model retains the essential aspects of the HF arbitrage activity: the intrinsic riskiness and the capability of providing liquidity in the market when needed, that is in response to cyclical changes in the liquidity positions of the market participants.

Having connected the repo markets, and in particular the analysis of the *collateral re-use*, to the DB business model, profitability and long term incentives, widens the set of available testable hypotheses in this field. In particular, it invites to empirically investigate whether changes in the setting of the policy parameter θ can affect differently the different financial systems, depending on the overall background business conditions. But it also enables to study the evolution in the equilibrium business models of the different DB (for instance, the amount of re-used collateral) in relation not only to the policy parameter θ (which in several jurisdictions has remained unchanged over the years), but also to the peers' bank specific business conditions.

VI. Conclusions

The work provides a model where the implications of *collateral re-use* in repo contracts are specifically focussed. The modelled economy enables to analyse both the borrowing and the lending side of the intermediation activity of a representative dealer bank. The provision of repo lending to a representative hedge fund takes place in an oligopoly regime. In this framework the two main profit maximising agents, the dealer bank and the hedge fund, select an optimal equilibrium for both the financial and the credit market. The selected equilibrium complies with regulatory constraints, among which a limitation in the *collateral re-use*. The main conclusions of the work are that in absence of default risk on the HF side, the increase in *collateral re-use* increases both the dealer bank and the hedge fund profits, improving the credit market conditions in terms of equilibrium conditions (rates and volumes), improving the financial market conditions in terms of volumes intermediated (liquidity), but also amplifying the leverage of the whole economy. Introducing a default risk on the side of the hedge fund, the proposed model allows to identify diverging effects of the *collateral re-use* on financial stability, depending on the background market conditions. In settings characterised by low dealer bank profitability, the increase in *collateral re-use*, despite increasing leverage, can improve the profitability of a prudent strategy on the side of the dealer bank, with the consequence of bringing to zero the correlation between hedge fund's and dealer bank's defaults, that is to say the risk of *contagion*. In settings where substantial additional profits can be produced by the bank trespassing the limits of a prudent credit provision, the increase in *collateral re-use* provides incentives to deviate from the sound management of the dealer bank. The work isolates new testable hypotheses, by providing a bridge between dealer bank profitability and business model and *collateral re-use* levels and policies.

Appendix A. DB problem solution

In order to solve the problem, we examine the possible cases defined by the equations (25-29):

1. All three constraints hold

If all three constraints hold then from equations 20, 21, and 19 we derive $r = m_r$, $D_{HF} = m_k$, and $m_k - \frac{m_r}{\phi} - m_d = 0$, respectively. This implies: $\frac{I(1-\lambda_h(1+r_i))}{1-\theta} = m_k - \frac{m_r}{\phi}$, identifying the following value of θ as equilibrium solution:

$$\theta_1 = 1 - \frac{I(1 - \lambda_h(1 + r_i))}{m_k - \frac{m_r}{\phi}} \tag{A1}$$

With $0 < \frac{I(1-\lambda_h(1+r_i))}{m_k - \frac{m_r}{\phi}} \leq 1$ as existence condition (in fact: $0 \leq \theta < 1$).

2. Case for $\delta > 0$, $\psi = 0$, and $\gamma > 0$

In this case we obtain from equations 21 and 19 the following: $r = m_r$ and $D_{HF} = \frac{r}{\phi} + m_d$. Since m_d is increasing in θ, D_{HF} in increasing in θ too. The equilibrium rate, on the contrary is stable at its maximum level (defined by the *participation constraint*).

From equation 25 we obtain $m_r - r_p - \delta = 0$, which leads to $\delta = m_r - r_p$. From equation 26 we obtain $\frac{m_r}{\phi} + m_d - 2\frac{m_r}{\phi} + \frac{r_p}{\phi} + \frac{\delta}{\phi} - \gamma = 0$. Substituting and simplifying we obtain $\gamma = m_d$, which indicates that the *shadow cost* of the *participation constraint* is increasing in θ. This is consistent with the expectations, since the profit of the HF is linearly increasing in the debt, which, in turns, is increasing in θ.

3. Case for $\delta > 0$, $\psi > 0$, and $\gamma = 0$

In this case, from equations 20 and 19 we obtain $D_{HF} = m_k$, $m_k - \frac{r}{\phi} - m_d = 0$, and, ultimately, to $r = \phi(m_k - m_d)$. In this case, being the solvency threshold already hit, the HF debt is no more increasing in the policy parameter θ. On the other hand, the equilibrium rate is decreasing in θ.

From equation 25 we obtain $\phi(m_k - m_d) - r_p - \delta - \psi = 0$. Expressing ψ explicitly, we obtain $\psi = \phi(m_k - m_d) - rp - \delta$. From equation 26 we obtain $m_k - 2(m_k - m_d) + \frac{r_p}{\phi} + \frac{\delta}{\phi} = 0$. Expressing δ explicitly, we obtain $\delta = 2\phi(m_k - m_d) - \phi m_k - r_p$.

4. Case for $\delta = 0$, $\psi > 0$, and $\gamma > 0$

This case is trivial and is not worth investigating in detail. In this case, in fact, being rich in liquidity, the DB is not in the condition of having to re-use the collateral obtained in the repo contracts *vis-à-vis* the HF. But this is only possible if either the deposits are too abundant (more than cover the financing necessities of the DB) or the liability side of the DB is expanded beyond efficient level. In both scenarios, this case does not fall within the scope of the analysis.

5. Case for $\delta > 0$, $\psi = 0$, and $\gamma = 0$

In this case, from equation 19 we obtain $D_{HF} - \frac{r}{\phi} - m_d = 0$ or, otherwise formulated, $r = \phi(D_{HF} - m_d)$.

From equation 25, we obtain $r - r_p - \delta = 0$ or $\delta = r - r_p$. Substituting into equation 26 we obtain $D_{HF} - 2\frac{r}{\phi} + \frac{r_p}{\phi} + \frac{r}{\phi} - \frac{r_p}{\phi} = 0$, which means that $r = \phi D_{HF}$.

The two results $r = \phi(D_{HF} - m_d)$ and $r = \phi D_{HF}$ imply that $m_d = 0$. The case of $m_d = 0$ is degenerate: it means that the DB has no resources to run its business (either $I = 0$ or $\lambda_h = 1$) and is outside of the scope of the analysis.

6. Case for $\delta = 0$, $\psi = 0$, and $\gamma > 0$

In this case, from equation 21 we obtain $r = m_r$.

From equation 25, we obtain $r = r_p$. But at this rate there is no DB participation. So this case falls outside of the analysis.

7. Case for $\delta = 0$, $\psi > 0$, and $\gamma = 0$

In this case, from equation 20 we obtain $D_{HF} = m_k$.

From equation 25, we obtain $r - r_p - \psi = 0$ or $\psi = r - r_p$. From equation 26, we obtain $m_k - 2\frac{r}{\phi} + \frac{r_p}{\phi} = 0$. Expressing r explicitly, we obtain $r = \frac{\phi m_k + r_p}{2}$.

The obtained solutions identify an area in which neither the rate nor the amount borrowed by the HF are affected by an increase in the policy parameter θ.

8. No constraint holds

This case is dominated by the others: if no boundary for rate, amount or liquidity limit is reached, since the profit function is increasing in both rate and amount, there is always a better solution to the problem on the boundary, which produces more profits for the DB than the internal ones.

The threshold θ_1 identifies the boundary value between the cases 2 and 3. We now need to identify critical value of θ that represents the boundary between the two cases 3 and 7. We are looking for the value of θ that makes $\delta = 0$. From case 3, we know that this happens when $2\phi(m_k - m_d) - \phi m_k - r_p = 0$. This requires that: $2\phi m_k - 2\phi\frac{I(1-\lambda_h(1+r_i))}{1-\theta} = \phi m_k + r_p$. This is verified when:

$$\theta_2 = 1 - \frac{2\phi I(1 - \lambda_h(1 + r_i))}{\phi m_k - r_p} \tag{A2}$$

Appendix B. DB problem solution - with HF default risk - "risky" strategy

In order to solve the problem, we examine the possible cases defined by the equations (47-51):

1. All three constraints hold

If all three constraints hold, then from equations 45, 46, and 44 we derive $D_{HF} = m_k$, $m_k - \frac{r}{\phi} - m_d = 0$, and $r = \frac{E[R]}{P} - 1 - \frac{m_p}{m_k}$, respectively. This implies: $r = \phi(m_k - m_d)$ and consequently $\phi(m_k - m_d) = \frac{E[R]}{P} - 1 - \frac{m_p}{m_k}$. Further derivations lead to $m_k - m_d = \frac{1}{\phi}\left[\frac{E[R]}{P} - 1 - \frac{m_p}{m_k}\right]$ and $\frac{I(1-\lambda_h(1+r_i))}{\theta-1} = \frac{1}{\phi}\left[\frac{E[R]}{P} - 1 - \frac{m_p}{m_k}\right] - m_k$ identifying the following value of θ as equilibrium solution:

$$\theta_1 = \frac{I(1 - \lambda_h(1 + r_i))}{\frac{1}{\phi}\left(\frac{E[R]}{P} - 1 - \frac{m_p}{m_k}\right) - m_k} + 1 \tag{B1}$$

With $-1 \leq \frac{I(1-\lambda_h(1+r_i))}{\frac{1}{\phi}\left[\frac{E[R]}{P}-1-\frac{m_p}{m_k}\right]-m_k} < 0$ as existence condition (in fact: $0 \leq \theta < 1$).

2. Case for $\delta > 0$, $\psi = 0$, and $\gamma > 0$

In this case we obtain from equations 44 and 46 the following: $D_{HF} - \frac{r}{\phi} - m_d$, $r = \phi(D_{HF} - m_d)$ and $r = \frac{E[R]}{P} - 1 - \frac{m_p}{D_{HF}}$. Since m_d is increasing in θ, D_{HF} in increasing in θ too. The equilibrium rate, on the contrary is an increasing function of the debt amount and a decreasing function of the policy parameter θ.

The equilibrium HF debt can be found following on in the derivation. From $\phi D_{HF} - \phi m_d = \frac{E[R]}{P} - 1 - \frac{m_p}{D_{HF}}$, the following second degree equation follows: $\phi D_{HF} + D_{HF}(1 - \frac{E[R]}{P} - \phi m_d) + m_p$. The highest root (the lowest is degenerate) delivers the equilibrium value for D_{HF}: $D_{HF} = \frac{\frac{E[R]}{P}+\phi m_d - 1 + \sqrt{(1-\frac{E[R]}{P}+\phi m_d)^2 - 4\phi m_p}}{2\phi}$

3. Case for $\delta > 0$, $\psi > 0$, and $\gamma = 0$

In this case, from equations 45 and 44 we obtain $D_{HF} = m_k$, $m_k - \frac{r}{\phi} - m_d = 0$. This leads to $r = \phi(m_k - m_d)$. In this case, being the solvency threshold already hit, the HF debt is no more increasing in the policy parameter θ. On the other hand, the equilibrium rate is decreasing in θ.

4. Case for $\delta = 0$, $\psi > 0$, and $\gamma > 0$

This case is trivial and is not worth investigating in detail.

5. Case for $\delta > 0$, $\psi = 0$, and $\gamma = 0$

This case is outside of the scope of the analysis.

6. Case for $\delta = 0$, $\psi = 0$, and $\gamma > 0$

This case falls outside of the analysis.

7. Case for $\delta = 0$, $\psi > 0$, and $\gamma = 0$

In this case, from equation 45 we obtain $D_{HF} = m_k$.

From equation 48, we obtain $m_k - 2\frac{r}{\phi} + \frac{r_p}{\phi} = 0$. Expressing r explicitly, we obtain $r = \frac{\phi m_k + r_p}{2}$.

The obtained solutions identify an area in which neither the rate nor the amount borrowed by the HF are affected by an increase in the policy parameter θ.

8. No constraint holds

This case is dominated by the others.

The threshold θ_1 identifies the boundary value between the cases 2 and 3. We now need to identify critical value of θ that represents the boundary between the two cases 3 and 7. But this threshold is the same as in the previous section (DB problem without default risk):

$$\theta_2 = 1 - \frac{2\phi I(1 - \lambda_h(1 + r_i))}{\phi m_k - r_p} \tag{B2}$$

Appendix C. DB problem solution - with HF default risk - "safe" strategy

In order to solve the problem, we examine the possible cases defined by the equations (55-59):

1. All three constraints hold

Same results as in the "risky" strategy scenario, including the definition of θ_1

2. Case for $\delta > 0$, $\psi = 0$, and $\gamma > 0$

Same results as in the "risky" strategy scenario.

3. Case for $\delta > 0$, $\psi > 0$, and $\gamma = 0$

Same results as in the "risky" strategy scenario.

4. Case for $\delta = 0$, $\psi > 0$, and $\gamma > 0$

This case is trivial and is not worth investigating in detail.

5. Case for $\delta > 0$, $\psi = 0$, and $\gamma = 0$

This case is outside of the scope of the analysis.

6. Case for $\delta = 0$, $\psi = 0$, and $\gamma > 0$

This case falls outside of the analysis.

7. Case for $\delta = 0$, $\psi > 0$, and $\gamma = 0$

In this case, from equation 45 we obtain $D_{HF} = m_k$.

From equation 56, we obtain $m_k - 2\frac{r}{\phi} + \frac{r_p}{\phi} + \frac{p}{\phi} = 0$. Expressing r explicitly, we obtain $r = \frac{\phi m_k + r_p + p}{2}$.

The obtained solutions identify an area in which neither the rate nor the amount borrowed by the HF are affected by an increase in the policy parameter θ. Nonetheless, in this case, the equilibrium value for the rate is higher, indicating a lower market share of the DB in comparison to the "risky" strategy scenario. In this scenario, the business strategy of the DB is less "aggressive".

8. No constraint holds

This case is dominated by the others.

The threshold θ_1 identifies the boundary value between the cases 2 and 3 and is equal to the one identified for the "risky" strategy scenario:

$$\theta_1 = \frac{I(1 - \lambda_h(1 + r_i))}{\frac{1}{\phi}\left(\frac{E[R]}{P} - 1 - \frac{m_p}{m_k}\right) - m_k} + 1 \tag{C1}$$

We now need to identify critical value of θ that represents the boundary between the two cases 3 and 7. This threshold is identified by comparing the equilibrium solutions for the rates in cases 3 and 7. More precisely, the θ that corresponds to the threshold between the two regimes is the θ that verifies $\phi(m_k - m_d) = \frac{\phi m_k + r_p + p}{2}$.

Which leads to:

$$\theta_3 = \frac{I(1 - \lambda_h(1 + r_i))}{\frac{1}{2\phi}(r_p + p - \phi m_k)} + 1 \tag{C2}$$

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