SAPIENZA
UNIVERSITÀ DI ROMA

ISSN 2385-2755
DiSSE Working papers
[online]

WORKING PAPERS SERIES
DIPARTIMENTO DI
SCIENZE SOCIALI ED ECONOMICHE

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N. 12/2020

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CF80209930587 – P.IVA 02133771002

How Low Interest Rates Discern the Bubbles Nature: Leveraged vs Unleveraged Bubble*

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11th June 2020

Abstract

Leveraged asset price bubbles, i.e., periods of boom-bust phases in asset prices accompanied by credit overhangs, are more harmful than unleveraged ones, in terms of financial and price stability. As bubbles are difficult to detect in real-time data, early researches focused on the macroeconomic conditions exacerbating the bubbles' nature. What kind of bubble is likely to emerge in an economy characterized by slow growth and a low real interest rate? This paper shows why the leveraged bubble is the answer to this question. First, we show that a negative real rate is sufficient for leveraged bubbles to emerge but not for unleveraged ones, in a stylized OLG model with incomplete credit markets and income inequality. Second, we show that this result holds empirically for post-World War II bubbles in advanced economies.

JEL Classification Numbers: E43, E44

Keywords: low interest rates, leveraged bubbles, unleveraged bubbles

*We are grateful to Luca Fornaro, Lenno Uuskula, Karsten Staehr and for their helpful comments.

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"However, not all asset price bubbles are alike.... In particular, some asset price bubbles can have more significant economic effects, and thus raise additional concerns for economic policymakers, by contributing to financial instability"

-Frederic Mishkin, Financial Stability Review no.12 2008, Banque de France

1 Introduction

Low risk-free interest rates are the hallmark of the post-2007 crisis era in many advanced economies. The downward trend of nominal and real interest rates observed in the economy is widely interpreted from the literature as a decline in the "natural" interest rate consistent with the potential output and stable prices (Rachel and Smith, 2015; Laubach and Williams, 2016; Holston et al., 2017). Although the historical decline started long before the Great Recession, it does not seem to arrest nowadays. Recent empirical evidence finds that pandemic shocks negatively affect the natural interest rate, envisaging low interest rates as a likely scenario of the years coming next to the pandemic Covid-19 (Jorda et al., 2020).

This paper investigates the theoretical and empirical implications of low *real* interest rates for the formation of asset price bubbles, to shed light on whether and to what extent *leveraged* and *unleveraged* bubbles are likely to emerge.

Persistent low risk-free interest rates expose the economy to financial instability, which can play out in several forms. Risk-taking behaviors and borrowing are encouraged because only risky investments are profitable and credit is cheap, fostering leveraged booms (Dell'Ariccia et al., 2014); while, if the real interest rate falls below the growth rate of the economy, asset price "bubbles" can emerge rationally (Baldwin and Teulings, 2014).[1]

A new asset price bubble, which would follow the boom-and-bust cycle in house prices that triggered the global financial crisis, seems particularly dangerous and highly detrimental.[2] The bursting of a new bubble would hurt economies that have only recently recovered the output losses suffered during the Great Recession. On the other hand, given the low interest rates, the central bank would not have enough space to cut policy rates further, and it could rely only on unconventional monetary policies to sustain the economy.

However, asset price bubbles are not inherently harmful. If they serve as a store of value without fostering credit growth, that is in the form of *unleveraged* bubbles, the economic cost of the bubble bursting is limited, and it does not turn necessarily in a financial crisis. *Leveraged* bubbles, in contrast, are accompanied by credit booms that can painfully hurt the economy, as they are more likely to trigger a financial crisis (Jordà et al., 2015b). Therefore, the distinction between leveraged and unleveraged bubbles is essential to study the potential threats that they pose to financial stability.

[1] An economic bubble can be defined as "the difference between market price and market fundamental"(Tirole, 1985).

[2] Low interest rates are a possible cause of the housing bubble itself (Taylor, 2014).

To investigate how low risk-free interest rates foster these two types of bubble, we first develop a theoretical model that studies the conditions under which they emerge. Second, we test the predictions of the theoretical model empirically, by exploiting a long historical dataset where leveraged and unleveraged bubbles can be identified for several countries.

Our theoretical framework is a two-period overlapping generations (OLG) model with income inequality, non-neutral monetary policy, rational bubbles, and an incomplete credit market. Income inequality shapes the characteristics of young households in the credit market, as low-income households are borrowers and high-income ones are lenders. When income is intensely concentrated among richer households, the natural rate of interest turns negative.[3] Moreover, if the inflation target is too low, the central bank cannot drive the real interest rate to its natural level via the monetary policy rate and the zero lower bound (ZLB) binds. Then, the economy gets stuck in an equilibrium characterized by low risk-free nominal and real interest rates.

In this economic environment, an intrinsically worthless asset ("bubble") can be valued by rational and optimizing agents, because it absorbs the excess of saving underlying a negative natural interest rate (Samuelson, 1958; Tirole, 1985). The bubble is (fully) unleveraged, when the lenders are the only owners of the bubbly asset, while it is (fully) leveraged when purchased exclusively by the borrowers. As the credit market is incomplete because of defaultable debt contract, low interest rates foster more a leveraged bubble than an unleveraged bubble.

Specifically, a leveraged bubble emerges if the real interest rate is lower than the economy's growth rate. This condition is always met in a *low* interest rates equilibrium featuring a *negative* real interest rate. In contrast, a negative real interest does not guarantee *per se* the emergence of an unleveraged bubble.

We corroborate this theoretical finding with an empirical analysis in a panel of 19 countries in the period 1945-2016. We exploit the macrohistory database by Jordà et al. (2015a) and follow a similar approach for the empirical investigation. In particular, we first identify bubbles by looking at the deviations from the long-run trend of both asset and house prices. Then, we differentiate leveraged from unleveraged bubbles by identifying those bubbly episodes that are concomitant with credit booms. The empirical model features the estimation of a logit function, where the dependent variable distinguishes the periods that anticipate the emergence of the two types of bubble. We focus on how the real interest rate predicts those states. Yet, we investigate the role that the real interest rate plays when moves slower than the growth rate of the economy, and when it is negative. The real interest rate alone plays a marginal role in distinguishing leveraged from unleveraged bubbles. However, during periods characterized by negative (low) real interest rates any further decrease in the real rate does sharply increases the probability that the coming bubble will be leveraged rather than unleveraged.

The remainder of the paper is organized as follows. Section 2 summarizes the related literature

[3] According to Eggertsson et al. (2019), the decline in the natural rate of interest is mainly caused by demographic and technological factors, not by income inequality. We do not replicate this specific causal nexus because our theoretical argument is invariant to the source of low interest rates. Instead, we use income inequality to deliver them and to distinguish between the two types of bubble through the identity of their owners.

to both the theoretical and the empirical parts. We present the theoretical model in Section 3, whereas we show the main features related to its steady state equilibrium in Section 4. Section 5 illustrates the empirical model and the main results. Section 6 presents concluding remarks.

2 Related Literature

The theoretical part of the paper is related to two strands of the literature. First, it is inspired by the recent literature on "secular stagnation" that views low interest rates as a result of the historical decline in the natural interest rate (Stiglitz, 2012; Summers, 2014, 2015; Baldwin and Teulings, 2014; Gordon, 2015; Eggertsson et al., 2019). In particular, Stiglitz (2012) argues that increasing income inequality puts downward pressure on interest rates and provides fertile ground for bubbles, but he does not formalize this idea. In contrast, the idea that inequality depresses the natural interest rate is formalized by Eggertsson et al. (2019), who develop a tractable OLG model to represent the main sources of low interest rates, without, however, investigating the emergence of asset price bubbles. We augment the theoretical framework of Eggertsson et al. (2019) with an incomplete credit market and rational asset price bubbles to study the different conditions under which leveraged and unleveraged bubbles arise in a low interest rates environment. Second, our work is linked to the theoretical literature on rational asset price bubbles in the OLG framework, which includes Samuelson (1958); Tirole (1985); Weil (1987); Martin and Ventura (2010, 2012); Bengui and Phan (2018), to name a few. As standard in this literature, we assume the source of the bubble is an excess of saving over investment, and the bubble can be either a store of value (Samuelson, 1958; Tirole, 1985) and a collateral (Martin and Ventura, 2012). Still, we enrich our OLG model with non-neutral monetary policy and a negative natural interest rate. Furthermore, we introduce leveraged and unleveraged bubbles along the lines of Bengui and Phan (2018), extending their results regarding the existence of the two types of bubble, limited to an endowment economy, to a production economy in which the interaction of a negative natural interest rate and the ZLB causes low interest rates.

The empirical part relates to the empirical literature about financial crises, started with Jordà et al. (2013) and followed by many other such as Jordà et al. (2015a,b). In general, they adopt a logistic framework to test the predictive power of credit dynamics to financial crises on a wide historical dataset. They find evidence that credit-driven asset price bubbles exacerbate both the risks of financial crises and subsequent output losses. Relative to those works, our analysis takes a different perspective as our dependent variables are bubble events rather than financial crises, while the real interest rate and other macro controls are the predictors. Moreover, as for the theoretical model, our empirical analysis is agnostic about the prediction and consequences of the bubble bursting.

At the same time, our approach is not that far from Jordà et al. (2015b). In that work, they focus on the relationship of leveraged and unleveraged bubbles with financial crises, and they find that the former is more costly than the latter in terms of financial stability and output deterioration. Our analysis shares the same spirit but with the opposite viewpoint. Our focus is indeed on the conditions that determine the occurrence of leveraged and unleveraged bubbles, that is the phases that precede the bubble formation, not on the bursting phase of the bubble and its consequences. For this reason, we do not study the magnitude and the duration of the bubbles, as Jordà et al. (2015b) do instead.

3 Model

We study a two-period OLG economy in which agents form expectations rationally and are perfectly informed. The size of the generations is constant and normalized to 1. Firms operate for one period and, as there is no capital, they employ only the labor input L_t. The production technology is given by

$$Y_t = L_t^{\alpha}, \tag{1}$$

where $0 < \alpha < 1$. As goods and labor markets are perfectly competitive, firms take the price of goods (P_t) and labor services (W_t) as given to maximize their profits $Z_t = P_t Y_t - W_t L_t$, and labor is remunerated at its marginal productivity:

$$\frac{W_t}{P_t} = \alpha L_t^{\alpha-1}. \tag{2}$$

We extend this standard OLG framework in three crucial dimensions.

Income inequality: Young households supply inelastically their labor endowment $\bar{L} = (1-\chi)\,\bar{L}^B + \chi \bar{L}^L$ and run firms. A share χ of the young households, which are *lenders*, have a high labor endowment $\bar{L}^L$ and a resulting high income to save, while the remaining share consists of *borrowers* who have a low labor endowment $\bar{L}^B$ and cannot save because of a low labor income.[4] Throughout the paper, the superscript L denotes lenders, and the superscript B denotes borrowers.

Markets for assets: Old households sell a bubbly asset, which has fixed unit supply, to young ones in a proper market. A "bubbly" asset has a fundamental value of zero, but it is purchased at a positive price when the buyer expects to resell it at a higher price. Each period the price of the bubble, $\tilde{p}_t^b$, can go to 0 with probability $\rho \in [0,1)$ and, if the bubble has already crashed, it never re-emerges (Weil, 1987). Conditional on not having collapsed, the price of the bubble is $\tilde{p}_t^b = p_t^b > 0$.

On the other hand, young borrowers sell a one-period bond to young lenders in the credit market. The credit market is incomplete because borrowers cannot commit to paying all their

[4] The higher total income of lenders ($Y_t^L > Y_t^B$) is due to higher labor income. On the one hand, the labor endowment of lenders is larger. On the other hand, the demand for their labor services is a constant share of the aggregate labor demand, $L_t^L/L_t = \bar{L}^L/\bar{L} = (1-\varepsilon)/(1-\chi)$, which equals the corresponding share of the total labor endowment. Therefore, lenders supply more labor and work more than borrowers, getting a higher labor income. Furthermore, high-income households could also borrow and default, but, if a fraction of their savingS can be seized, the optimal borrowing level is zero. This result is shown in a similar setting by Bengui and Phan (2018).

outstanding debt, but they issue a non-contingent standard debt contract, which is defaultable and whose gross real interest rate charged on, $(1 + r_t)$, does not depend on the size of the loan (Allen and Gale, 2000; Ikeda and Phan, 2016).

Downward nominal wage rigidity and non-neutral monetary policy: Workers, borrowers and lenders, are unwilling to accept a nominal wage below a minimum level:

$$W_t = \max\left\{\gamma\Pi^* W_{t-1}, \alpha P_t \bar{L}^{\alpha-1}\right\}, \tag{3}$$

where $\gamma \in (0,1)$ and $\Pi^* > \gamma\Pi^* > 1$. The lower bound on the nominal wage, $\gamma\Pi^* W_{t-1}$, is a fraction of its past level indexed to the gross inflation target Π^*, while $\alpha P_t \bar{L}^{\alpha-1}$ is the "flexible" wage corresponding to full employment. The downward nominal wage rigidity (DNWR) allows for the non-neutrality of monetary policy,[5] which is specified in terms of the standard Taylor rule

$$1 + i_t = \max\left(1, \left(1 + r^f\right)\Pi^* \left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi}\right), \tag{4}$$

where $\phi_\pi > 1$, $\left(1 + r^f\right)\Pi^*$ is the target for the gross nominal interest rate and r^f is the "natural" rate of interest corresponding to output at the potential level, $Y^f = \bar{L}^\alpha$. Finally, the standard Fisher equation

$$1 + r_t = (1 + i_t)\, E_t \Pi_{t+1}^{-1} \tag{5}$$

holds, where $\Pi_t = P_t / P_{t-1}$ is the gross inflation rate and E_t is the expectation operator.

In this section, we outline the maximization problem of borrowers and lenders, along with the functioning of asset markets. This part is the core of our theoretical framework, while the supply-side of the model and monetary policy do not play any role as long as the DNWR and the ZLB are not at work. We postpone the explanation of remaining elements of the model to the next section regarding a steady state equilibrium with binding DNWR and ZLB.

3.1 Borrowers and Lenders

Households have logarithmic preferences and their consumption in the two stages of life is $C^i_{y,t}$ and $C^i_{o,t+1}$, where $i \in \{B, L\}$. At young age, borrowers get the income Y^B_t and pay a lump-sum tax T to finance social security benefits in old age. The borrowers' problem is

$$\max_{b^B_t \geq 0} E_t \left(\ln C^B_{y,t} + \beta \ln C^B_{o,t+1}\right)$$

s.t.

$$C^B_{y,t} = Y^B_t + d^B_t - \tilde{p}^b_t b^B_t - T$$

$$C^B_{o,t+1} = T + \tilde{p}^b_{t+1} b^B_t - (1 - \xi_{t+1})(1 + r_t)\, d^B_t - \xi_{t+1}\left(D + \phi \tilde{p}^b_{t+1} b^B_t\right)$$

[5] The model would be unchanged, if we assume Calvo pricing (Calvo, 1983). Eggertsson et al. (2019), whose theoretical model shares with ours the supply-side, prove formally this result.

$$(1+r_t)\, d_t^B = D + \phi p_{t+1}^b b_t^B.$$

β is the subjective discount factor and $Y_t^B = \frac{Z_t}{P_t} + \frac{W_t}{P_t} L_t^B$.[6] Borrowers cannot choose the level of borrowing d_t^B because they are credit constrained, and their borrowing is limited to the maximum amount that lenders can repossess in case of default, namely a "fundamental" collateral, $D \in (0, T)$, and a fraction ϕ of the borrowers' bubble holdings (Bengui and Phan, 2018).[7] As a consequence, the bubbly asset has a twofold role for borrowers: a collateral, which allows to collect additional resources, $\phi p_{t+1}^b b_t^B$, to consume today, and a store of value, which allows to carry over resources, $\tilde{p}_{t+1}^b b_t^B$, to consume tomorrow. Borrowers choose the optimal amount of bubble holdings b_t^B, which can be positive or zero, anticipating their decision to default or not when old. The default decision at time $t+1$ is governed by the rule:

$$\xi_{t+1} = \begin{cases} 0 & if \quad (1+r_t) d_t^B \leq D + \phi \tilde{p}_{t+1}^b b_t^B \\ 1 & if \quad (1+r_t) d_t^B > D + \phi \tilde{p}_{t+1}^b b_t^B. \end{cases} \tag{6}$$

If repaying is at least as convenient as defaulting, borrowers repay all their outstanding debt and ξ_{t+1} is zero. On the contrary, borrowers go bankrupt when defaulting is the most convenient option and ξ_{t+1} equals one. The borrowing limit, combined with the rule (6), implies that borrowers default only if the bubble bursts, namely if $\tilde{p}_{t+1}^b = 0$.

Instead, the maximization problem of lenders is

$$\max_{d_t^L, b_t^L \geq 0} E_t \left(\ln C_{y,t}^L + \beta \ln C_{o,t+1}^L \right)$$

s.t.

$$C_{y,t}^L = Y_t^L - d_t^L - \tilde{p}_t^b b_t^L$$

$$C_{o,t+1}^L = \tilde{p}_{t+1}^b b_t^L + (1 - h_{t+1})(1+r_t)\, d_t^L.$$

Lenders get a sufficiently high income, $Y_t^L = \frac{Z_t}{P_t} + \frac{W_t}{P_t} L_t^L$, to save, and they choose either the optimal bond purchases d_t^L and bubble purchases b_t^L. However, as lenders do not take on debt, they can use the bubbly asset, if purchased, only as a store of value. Furthermore, they can repossess a share of their original claims if borrowers default, and the remaining fraction of

[6] Given the DNWR (3), the demand for the labor services of borrowers L_t^B can be lower than or equal to their supply $\bar{L}^B$. A similar argument applies also to the demand for the labor services of lenders, L_t^L. Here, we are incorporating the labor rationing approach (Schmitt-Grohé and Uribe, 2016), which will be illustrated in the next section along with the DNWR.

[7] We impose

$$D < \frac{T}{1+\beta} - \frac{\beta}{1+\beta}(1+r_t)\left(Y_t^B - T\right)$$

and

$$D < \frac{T}{[1+\beta(1-\rho)]} - \frac{\beta(1-\rho)}{1+\beta(1-\rho)}\left[(1+r_t)\left(Y_t^B - T - p_t^b b_t^B\right) + p_{t+1}^b b_t^B\right]$$

so that borrowers are credit constrained either in a bubbleless economy and in a bubbly one with full collateralization of the bubble ($\phi = 1$). In the event of default, lenders repossess effectively the maximum possible amount only if the bubble survives, $\tilde{p}_{t+1}^b = p_{t+1}^b > 0$. Indeed, the borrowing constraint could be alternatively expressed as $(1+r_t)\, d_t^B = D + \phi max\{\tilde{p}_{t+1}^b\} b_t^B$.

losses on loans is the haircut h_{t+1}, which is a random variable:

$$h_{t+1} = \begin{cases} 0 & if \quad \xi_{t+1} = 0 \\ 1 - \frac{(1-\chi)}{\chi} \frac{D}{(1+r_t) d_t^L} & if \quad \xi_{t+1} = 1. \end{cases} \tag{7}$$

If there is no default, the haircut is zero. Instead, if borrowers pledge bubbly assets and then default, the aggregate fundamental collateral $(1-\chi)D$, which is a fraction of the total claims $\chi(1+r_t)\, d_t^L$, is distributed evenly to lenders, and the remaining fraction of the outstanding debt represents the haircut on loans.

The maximization problem of the two households will be solved in Section 4, where we study separately a bubbleless economy and a bubbly one. However, the crucial takeaway from this section is that lenders and borrowers have a different motive to hold the bubble. Lenders need an alternative store of value when there are few investment opportunities, while borrowers hold bubbly assets mainly because of their collateral value, which depends on their degree of pledgeability ϕ, that is the percentage of the bubble value that turns into credit. When bubbles are highly pledgeable, a high percentage of their value turns into credit and borrowers buy them to collect extra funds. As bubbles foster credit in this case, they are *leveraged* if borrowers partially or fully hold them, and they are *unleveraged* if lenders buy at least a fraction of the bubbly assets.

4 Equilibrium

Competitive equilibrium: Given W_{-1}, d_{-1}^L and $p_0^b \geq 0$, a competitive equilibrium consists of the prices $\{P_t, W_t, r_t, i_t, p_t^b\}$, the quantities $\{d_t^L, b_t^L, d_t^B, b_t^B, C_{y,t}^L, C_{o,t}^L, C_{y,t}^B, C_{o,t}^B, Y_t, Z_t, L_t, L_t^L, L_t^B\}$, the haircut h_{t+1} and the default decision ξ_{t+1} such that:

- households maximize their lifetime utility and firms maximize their profit;
- $Y_t = (1-\chi)\left(C_{y,t}^B + C_{o,t}^B\right) + \chi\left(C_{y,t}^L + C_{o,t}^L\right)$ (*goods market clears*)
- $L_t = \bar{L}$ for $W_t = \alpha P_t \bar{L}^{\alpha-1}$ or $L_t < \bar{L}$ for $W_t = \gamma\Pi^* W_{t-1}$ (*labor market clears or labor rationing*)
- $(1-\chi)\, d_t^B = \chi d_t^L$ and $\chi b_t^L + (1-\chi)\, b_t^B = 1$ if $p_t^b > 0$ (*markets for assets clear*)
- monetary policy follows the rule (4), equation (5) holds and h_{t+1} satisfies (7)

A particular feature of the equilibrium is the functioning of the labor market, which does not necessarily clear because of the DNWR (3). If market clearing requires an increase in W_t from the previous period of $\gamma\Pi^*$ or more, the nominal wage equals its flexible level and the labor market clears ($L_t = \bar{L}$). On the contrary, if an increase of less than $\gamma\Pi^*$ is necessary to clear the labor market, $W_t = \gamma\Pi^* W_{t-1}$ and involuntary unemployment arises ($L_t < \bar{L}$).[8]

[8]Any fall in the aggregate labor demand relative to the economy's labor endowment causes a proportional decline in the demand for borrowers and lenders' labor services, without redistributing income among young

The equilibrium just outlined is bubbleless for $p_0^b = 0$, while it is bubbly for $p_0^b > 0$. In this section, we will focus on bubbleless and bubbly steady state equilibria in which the variables take a constant value, so we remove the time subscript. First, we will study a bubbleless steady state that features binding ZLB and replicates the current low interest rates environment characterizing most of the advanced economies. Then, we will investigate how unleveraged and leveraged bubbly equilibria arise, starting from the bubbleless equilibrium with persistently low interest rates. The transitional dynamics is trivial because the economy reaches the bubbly equilibrium, which is the only asymptotic bubbly equilibrium ($\lim_{t\to\infty} p_t^b > 0$), immediately.

4.1 A Bubbleless Economy

Before analyzing the steady state equilibrium of the economy, we define the real interest rate that clears the credit market because it plays a crucial role in determining the general equilibrium. We start from the maximization problem of borrowers and lenders for $p^b = 0$. Borrowers are credit constrained:

$$d^B = \frac{D}{1+r}. \tag{8}$$

As regards lenders, the optimality condition for their maximization problem is the standard Euler equation

$$\frac{1}{C_y^L} = \beta\left(1+r\right)\frac{1}{C_o^L},$$

given that there is no default and $h = 0$. Combining the Euler equation and the two budget constraints yields the credit supply

$$d^L = \frac{\beta}{1+\beta} Y^L. \tag{9}$$

The market for credit clears at the equilibrium real interest rate

$$\left(1+r_{nb}\right) = \frac{\left(1-\chi\right)}{\chi}\frac{\left(1+\beta\right)}{\beta}\frac{D}{Y^L}, \tag{10}$$

which equates the demand for credit from borrowers and the supply from lenders. The subscript nb denotes a no-bubble economy. Equation (10) tells that a large share of the total income for lenders, χY^L, results in a negative real interest rate $(1 + r_{nb} < 1)$[9] and so in a negative natural rate of interest, which clears the credit market at $Y = Y^f$:

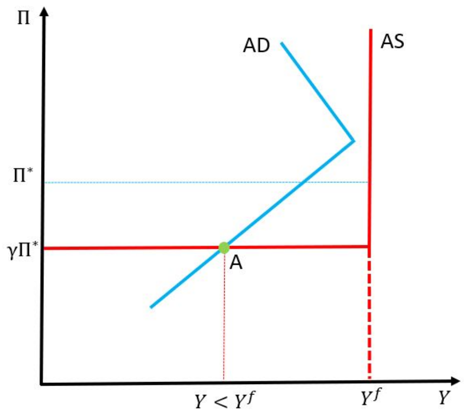
$$\left(1+r_{nb}^f\right) = \frac{\left(1-\chi\right)}{\chi}\frac{\left(1+\beta\right)}{\beta}\frac{D}{Y^{f,L}}.$$

$\chi Y^{f,L}$ is the fraction of the potential output attributed to lenders.

households. This results from the assumption that the demand for the labor services of lenders (and borrowers) is a constant share of the aggregate labor demand, and it corresponds to the lenders' (borrowers') share of the aggregate labor endowment.

[9] Although the income of lenders is endogenously determined by output, χY^L is a constant share of Y because of the assumption that the demand for the labor services of lenders and borrowers is constant share of the aggregate labor demand.

Figure 1: LIR Equilibrium in a Bubbleless Economy



The steady state equilibrium can be expressed by aggregate supply and demand, which are both characterized by two regimes. The regime of supply depends on the DNWR (3). For $\Pi \geq \gamma\Pi^*$, $W = \alpha P \bar{L}^{\alpha-1}$ and the aggregate supply (AS) corresponding to potential output,

$$Y_{AS} = \bar{L}^{\alpha} = Y^f, \tag{11}$$

can be computed from (1), (2) and (3). If the inflation rate is sufficiently high relative to the lower bound on wage and price inflation imposed by the DNWR, $\gamma\Pi^*$, the nominal wage is flexible and the economy runs at its potential level. On the contrary, if the flexible nominal wage is lower than the minimum wage level in (3), wage and price inflation is given by

$$\Pi = \gamma\Pi^*, \tag{12}$$

for any $L \leq \bar{L}$. In the case of binding DNWR, the level of output and employment is accordingly demand-determined. The AS curve in equation (11), the vertical segment, and equation (12), the horizontal segment, are both depicted in Figure 1 as a solid red line. The regime of aggregate demand (AD) depends on whether or not the ZLB binds according to the interest rule (4). When the nominal interest rate is positive, $1+i > 1$, the following AD can be derived from the equations (4), (5) and (10):

$$Y_{AD} = (1-\chi)\, Y^B + (1-\chi)\left(\frac{1+\beta}{\beta}\right)\left(\frac{\Pi^*}{\Pi}\right)^{\phi_\pi - 1}\frac{D}{1+r^f}. \tag{13}$$

Combining the same equations yields a different AD with a binding ZLB, $1+i=1$:

$$Y_{AD} = (1-\chi)\, Y^B + (1-\chi)\left(\frac{1+\beta}{\beta}\right)\Pi D. \tag{14}$$

Equation (13) expresses a negative relationship between inflation and output, which corresponds to the downward-sloping segment in blue depicted in Figure 1. This relationship turns positive at the ZLB, as shown by the upward-sloping segment of the AD curve in the same figure. Far away from the ZLB, the central bank tracks the natural interest rate, but it reacts to higher inflation by raising the policy rate more than proportionally ($\phi_\pi > 1$). This contracts demand and stabilizes inflation around the targeted level. At the ZLB, standard monetary policy tools do not allow the central bank to equate the real interest rate to its natural level. Therefore, the real interest rate is determined exclusively by the inflation rate in the Fisher equation (5), and when inflation rises, the real rate falls and demand increases.

The relationship between $1+r^f_{nb}$ and Π^* is crucial to determine the nature of the steady state equilibrium because it governs the regime of monetary policy in equation (4). When $1+r^f_{nb} < 1$ due to the presence of income inequality and, specifically,

$$1+r^f_{nb} < \frac{1}{\Pi^*} < 1, \tag{15}$$

the ZLB constrains the monetary policy. The inflation target, though positive ($\Pi^* > 1$), is not high enough to drive the real interest rate to its negative natural level trough standard monetary policy tools. Yet, the central bank cannot set a positive policy rate, and a binding ZLB implies low real and nominal risk-free interest rates. The equilibrium described corresponds to point A in Figure 1. This low interest rates (LIR) equilibrium features a negative real interest rate, binding ZLB, the output below the potential and inflation positive but below the target:[10]

$$1+r^f_{nb} < 1+r_{nb} < 1$$

$$i = 0$$

$$Y < Y^f$$

$$1 < \Pi = \gamma\Pi^* < \Pi^*.$$

Such kind of equilibrium is far from being unrealistic and replicates what is currently observed in the aftermath of the pandemic Covid-19 for the large majority of the developed economies.[11]

[10] All the steady state variables are shown in Appendix A.1, while, in Appendix A.2, we show that the existence and the nature of the LIR equilibrium is unaffected qualitatively by assuming that the DNWR depends on the level of employment (Eggertsson et al., 2019). Finally, the LIR equilibrium is determinate, because the determinacy requires $1-\alpha < 1$. A formal derivation of the condition for determinacy is available upon request.

[11] For $1+r^f_{nb} < 1$, condition (15) guarantees the unique equilibrium is the LIR one just outlined. However, if (15) does not hold and $\Pi^* > 1/\gamma(1+r^f)$, the unique equilibrium is different from that one because it does not feature a binding ZLB, despite a negative natural (=real) interest rate (Ascari and Bonchi, 2019). The theoretical findings regarding the emergence of leveraged and unleveraged bubbles hold also considering this alternative equilibrium, given that a negative real interest rate is crucial for our results, not a binding ZLB. Notwithstanding, the alternative equilibrium features inflation at the target that is inconsistent with the current evidence of the advanced economies, likewise a positive policy rate.

4.2 A Bubbly Economy

We restrict the analysis of the bubbly equilibrium to the cases of fully unleveraged and fully leveraged bubbles. By allowing only these two types of bubble in the model, we can reconcile the theoretical implications with the data. From an empirical viewpoint, “mixed” bubbles are indeed hard to identify and fairly infrequent. We clarify this point in the next Section 5.

4.2.1 Unleveraged Bubble

A fully unleveraged bubble can arise for $\phi = 0$. As bubbly assets cannot be collateralized, the bubble is unleveraged by construction. Furthermore, borrowers cannot borrow against the bubble and they have a weak incentive to hold it. Therefore, only lenders are allowed to invest in the bubble by assumption, so that $b^B = 0$ and $b^L = 1/\chi$.[12] In this case, there is no default risk, $\xi = h = 0$, and the borrowers' are still credit constrained (8) as in the bubbleless economy. For the lenders' problem, the budget constraints become

$$C_y^L = Y^L - d^L - p^b b^L$$

$$C_o^L = \begin{cases} (1+r)\, d^L & bubble \quad bursts \\ p^b b^L + (1+r)\, d^L & bubble \quad survives, \end{cases}$$

while the resulting optimality conditions that express the choice of bubbly assets and lending are

$$\frac{1}{C_y^L} p^b = \beta (1-\rho) \left[\frac{1}{p^b b^L + (1+r)\, d^L} \right] p^b \tag{16}$$

$$\frac{1}{C_y^L} = \beta (1+r) \left[\rho \frac{1}{(1+r)\, d^L} + (1-\rho) \frac{1}{p^b b^L + (1+r)\, d^L} \right]. \tag{17}$$

When the bubble is fully unleveraged, lenders bear completely the risk of bursting, and their consumption level in old age varies depending on whether the bubble bursts or not. From equations (8), (16), (17), and the credit market clearing condition, we get the equilibrium price of the bubble:

$$p^b = \frac{1}{b^L} \left[(1-\rho) \frac{\beta}{1+\beta} Y^L - \frac{(1-\chi)}{\chi} D \right].$$

The condition for the existence of an unleveraged bubble, $p^b > 0$, is then:

$$1 - \rho > 1 + r_{nb} = \frac{(1-\chi)}{\chi} \frac{(1+\beta)\, D}{\beta Y^L}. \tag{18}$$

Therefore, we can state the following proposition:

[12] As the bubbly asset is complex security that is both a store of value and collateral for borrowers, they could hold bubbly assets for $\phi = 0$. We study the case in which borrowers hold a small fraction of the bubble in Appendix A.3. However, $b^B > 0$ does not alter the nature of the bubbly equilibrium that is fully unleveraged for $\phi = 0$.

Proposition 1. *Assume $\phi = 0$, $b^B = 0$ and $b^L = 1/\chi$, then a fully unleveraged bubble exists only if the gross real interest rate prevailing in a bubbleless economy, $1 + r_{nb}$, is lower than the probability that the bubble survives, $1 - \rho$. Therefore, a negative real interest rate is necessary, but not sufficient, condition for the existence of the fully unleveraged bubble.*

An excess of saving, due to income inequality, drives both the natural and the real interest rates in negative territory and the economy is stuck in an LIR equilibrium. When there is no sufficient store of value in the bubbleless economy, lenders will buy intrinsically worthless assets if their return, $1 - \rho$, is higher than bonds' return, $1 + r_{nb}$. This condition has two crucial implications. First, as lenders invest their income and bear the bubble bursting risk, they will value the bubble as long as its survival probability is sufficiently high. Consequently, an extremely risky unleveraged bubble, associated with a too high probability of bursting, is not possible. Second, and related to this point, the negative real interest rate, $1 + r_{nb} < 1$, prevailing in the LIR equilibrium is not sufficient for the existence of an unleveraged bubble because it does not make necessarily the bubble more profitable than bonds. This result extends to the case of a positive economy's growth rate, $g > 0$, whose corresponding condition for the existence of the unleveraged bubble is $(1 - \rho)(1 + g) > 1 + r_{nb}$. Indeed, a sufficiently high probability of bubble bursting, $\rho > \frac{g - r_{nb}}{1 - g}$, and/or an excessively low economy's growth rate, $g < \frac{r_{nb} + \rho}{1 - \rho}$ for $\rho > -r_{nb}$, can prevent the emergence of the bubble even in this case, despite the negative real interest rate.

4.2.2 Leveraged Bubble

A *fully* leveraged bubble exists only if $\phi = 1$. Borrowers will find bubbles extremely attractive if these are wholly pledged in the credit market, and their demand fulfills all the supply in this case, that is $b^B = 1/(1 - \chi)$ and $b^L = 0$. The bubble is then *leveraged* because it is used as collateral by borrowers. On the other hand, lenders invest all their savings in bonds, which guarantee a higher return than bubbles and are risky assets, unlike the unleveraged case, because guaranteed by borrowers' bubble holdings.[13]

The budget constraints of lenders are different from the unleveraged case because $h > 0$ and $b^L = 0$. Lenders' optimal condition in equation (16) holds now with inequality, whereas the one in equation (17) reduces to the Euler equation derived in the bubbleless case, because of $b^L = 0$. For borrowers, the constraints of their maximization problem are now

$$C_y^B = Y^B + d^B - p^b b^B - T$$

$$C_o^B = \begin{cases} T - D & \textit{bubble} \quad \textit{bursts} \\ T + p^b b^B - (1 + r)\, d^B & \textit{bubble} \quad \textit{survives} \end{cases}$$

$$d^B (1 + r) = D + p^b b^B,$$

[13] For $\phi = 1$, if borrowers hold the bubble, lenders optimally choose to do not. A formal proof of that is given in Appendix A.4.

Borrowers' optimal conditions are then:

$$\frac{1}{C_y^B}p^b = \beta(1-\rho)\left[\frac{1}{T + p^b b^B - (1+r)\, d^B}\right]p^b + \lambda_d^B p^b \tag{19}$$

$$\lambda_d^B = \frac{1}{C_y^B(1+r)} - \beta(1-\rho)\frac{1}{T + p^b b^B - (1+r)\, d^B} > 0, \tag{20}$$

where λ_d^B is the Lagrange multiplier associated with the credit constraint. The latter is strictly positive because borrowers are credit constrained even if the bubble can be fully collateralized:

$$d^B = \frac{D + p^b b^B}{1+r}. \tag{21}$$

Leveraged bubble features *risk-shifting*. If borrowers pledge bubbly assets to collect additional resources to consume, they choose to default in case of bubble bursting and repay only the fundamental collateral. Therefore, by borrowing against the bubble, borrowers shift the downside-risk of the bubbly investment to lenders. Risk-shifting affects the condition of the leveraged bubble to existing. The equilibrium price of a fully leveraged bubble,

$$p^b = \chi\left[\frac{\beta}{1+\beta}Y^L - \frac{(1-\chi)}{\chi}D\right],$$

can be obtained through equations (9), (19), (20), (21), and the credit market clearing condition. p^b is positive and the bubble exists if

$$1 > 1 + r_{nb} = \frac{(1-\chi)}{\chi}\frac{(1+\beta)\, D}{\beta Y^L}. \tag{22}$$

This leads us to state the following proposition:

Proposition 2. *Assume $\phi = 1$, $b^B = 1/(1-\chi)$ and $b^L = 0$, then a fully leveraged bubble exists only if the gross real interest rate prevailing in a bubbleless economy, $1 + r_{nb}$, is lower than one. Therefore, a negative real interest rate is a sufficient and necessary condition for the existence of the fully leveraged bubble.*

From an economic viewpoint, equations (22) and (18) can be interpreted in the same way. Rational bubbles can emerge if the bubbleless economy lacks sufficient investment opportunities so that the supply of savings exceeds the demand for borrowing. However, unlike in the equation (18), the probability of bursting does not enter equation (22)because of risk-shifting like in Bengui and Phan (2018). As borrowers do not invest their income, they do not internalize the risk of bubble collapse, and bubbly assets no longer need to have a sufficiently high probability of surviving to be profitable, and so to be valued. As a consequence of this, leveraged bubbles are generally riskier than unleveraged ones.

Furthermore, a negative real interest rate, which generally prevails in an LIR equilibrium, is sufficient for a leveraged bubble to emerge because it always makes the bubble profitable for borrowers. This last result also applies in the case of a positive economy's growth rate, in which the existence of a leveraged bubble requires $1 + g > 1 + r_{nb}$.

5 The empirical prediction of the bubbles' nature

Our theoretical model shows that a low interest rates environment featuring a negative real interest rate is more prone to favor leveraged bubbly episodes than unleveraged ones. This result comes from the fact that an economy's growth rate higher than the risk-free real interest rate is a sufficient condition for leveraged bubbles to emerge but not for unleveraged bubbles. We test this theoretical implication empirically by employing a long-run macro dataset –i.e., the JST Macrohistory Database (Jordà et al., 2013, 2015a,b), – in a logit model estimation. We aim at testing the power of the real interest in distinguishing leveraged bubbly episodes from unleveraged ones. As in our OLG model, we are agnostic about the prediction and the consequences of the bubble bursting, and we focus on the LIR equilibrium as a proactive source for bubbles' emergence. Our analysis covers a sample of annual data for 17 advanced economies from 1945 to 2016.[14] Specifically, we use a novel data-release that includes asset price dynamics and that can be retrieved from Jordà et al. (2019).[15]

We present first the variables involved in the estimation, i.e., the real rate of interest and the bubbles identification. Then, we illustrate the panel logit model and discuss the results.

5.1 The real interest rate and the growth rate of the economy

The real rate of interest is the core variable of our empirical analysis. Consistently with the theoretical model, we define this risk-free rate as a short-term return on safe assets, and we use the 3-months government bond yields as the nominal rate of our benchmark specification.[16] To check the robustness of our results, we also run our empirical model with a long-term return on safe assets, which is less volatile and less influenced by cyclical components, and with other types of rate of returns.

We estimate the "ex-ante" real rate of interest *via* the Fisher equation (5). We embrace the standard empirical approach that the real rate should account for time variation in inflation persistence (Hamilton et al., 2016; Borio et al., 2017; Lunsford and West, 2019). Specifically, we proxy expected inflation by recursively projecting an autoregressive process, AR(1), estimated over a rolling 20-year window. Then, we subtract expected inflation from the short-term nominal interest rate.

[14] The countries are Austria, Belgium, Canada, Denmark, Finland, France, Germany, Italy, Japan, the Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, the UK, and the US.

[15] Table 3 of Appendix B summarizes the data used in the analysis.

[16] The OLG structure allows for a tractable analysis of low interest rates and rational bubbles, but birth/death should be interpreted as the entry/exit of the agents in the credit market, as usual in the financial friction literature (e.g.,Bernanke and Gertler, 1989). Therefore, a period is the length of a loan contract rather than that of a generation.

Figure 2 reports the cross-country median of the real interest rate in level (blue line), the real GDP growth rate (red line) and two percentile intervals in the grey areas – 35th-65th (30%) and 20th-80th (60%) – showing the dispersion of the time-series among countries. The graph starts in the post-World War II period. During the decade 1960-1970, the countries' real rates were steady and the economic growth sustained, until the simultaneous drop, corresponding to the "Great Inflation" period of the mid-1970s. Then, the real rate climbed up until the early-1990s, while the economic growth was moving in the grace period of "Great Moderation" between the mid-1990s and the early-2000s. From then on, the real rate slowed down gradually up to nowadays, except for the "Great Recession" period in which the rapid increase was mostly due to the tensions in the European sovereign debt markets.

The condition $r \leq g$ holds in 45% of observations in our sample, while $r \leq 0$ in 13% of them. Both conditions hold mainly in the years of the Great Inflation and the Great Recession. The frequency of negative real rates increases progressively and substantially, starting from 1995 up to the end of the sample (see Figure 4 of Appendix B). As shown in the next subsection, this period has the highest bubbles' concentration, mostly due to the so-called "dot-com bubble" in 2000, and the housing bubble in 2005 in many advanced economies.

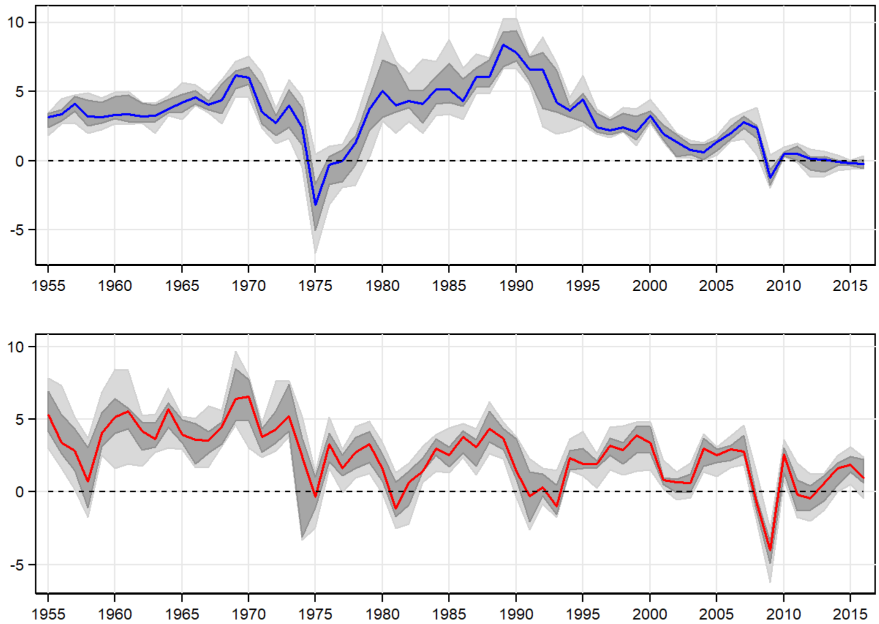


Figure 2: Cross-country medians of real interest rate (blue line) and annual growth of real GDP (red line). The bands show different percentile intervals, 35th-65th (30%), 20th-80th (60%) and 5th-95th (90%).

5.2 Bubbles identification: leveraged and unleveraged

We employ a systematic strategy to catch bubbly events in our sample. First, we identify generic bubbles by looking at both real, CPI-deflated, equity and house prices. Equity prices and house prices are indexed by taking 1990 as the reference year.[17] Second, to distinguish leveraged from unleveraged bubbles, we look at overhangs of total loans to non-financial private sector relative to the country's GDP.

We identify the buildup of an asset price bubble and a credit-to-GDP overhang by qualifying a significant increase in the cyclical component as a deviation from its rolling-window standard deviation. Therefore, we consider a bubble any displacement from the long-run trend higher than the cyclical-standard deviation. This procedure reflects the theoretical intuition of having price deviations from the fundamental value of an asset.

We employ the Hamilton filter (Hamilton, 2018), to get the cyclical components of credit-to-GDP, real equity prices, and real house prices. This filter regresses an actual observation on its past observations for a given time horizon, so that the cyclical component is just the residual of the regression. In this way, the long-run trend is conceived as what can be explained using historical data.[18] We projects an horizon of 20 years, which is consistent with the long duration of financial cycles (Hamilton, 2018; Drehmann and Yetman, 2018), by estimating the following equation through OLS:

$$y_{t+h} = \beta_0 + \beta_1 y_t + \beta_2 y_{t-1} + \beta_3 y_{t-2} + ... + \beta_{19} y_{t-20} + \epsilon_{t+h}. \tag{23}$$

A boom buildup is identified whenever the cyclical component, ϵ_t, passes from being $\epsilon_t < \sigma(\epsilon_t)$ to $\epsilon_t \geq \sigma(\epsilon_t)$, where $\sigma(\epsilon_t)$ is 20-years-window standard deviation of the cyclical component. This boom condition identifies also credit overhang periods.

However, we identify an asset price bubble only when a boom is followed by a price burst. As in Jordà et al. (2015b), the burst is defined as a decline in the (cyclical) asset price of at least 15% (a change of 0.15 log-points) within the three years from any point in which the boom condition holds. This further requirement completes our empirical definition of bubble because it rules out all those price expansions that are rational adjustment to changes in fundamentals.

A leveraged bubble is identified wherever the bubble condition holds together with a credit-to-GDP overhang at any point in the time of the bubble event. When a bubble occurs during a period in which credit-to-GDP is not that far from its long-run trend, we identify it as an unleveraged bubble. Figures 5-6 show the bubbles identified for selected countries. For each country, we report two panels: the upper one shows the real equity prices cycle (blue line), and the bottom one shows the real house prices cycle (red line). Both panels report the relative asset price threshold (dotted lines), the credit cycles (black dashed lines), and the relative leveraged bubbles

[17] Equity prices are the total return on all stocks listed on the country's stock exchange and market cap weighted.

[18] Drehmann and Yetman (2018) find that the de-trended credit-to-GDP ratio obtained through one-sided Hodrick-Prescott (HP) filter, with a higher smoothing parameter λ, outperforms many other measures of credit gap in predicting financial crises. The identification through the one-side HP filter delivers similar bubbly events to the Hamilton filter and the results are available upon request.

(red bars) and unleveraged bubbles (grey bars). The graphs show how this strategy catches some of the most famous bubbles of the recent countries' history, such as, for instance, the asset price bubble in Japan (1989) and the dot-com bubble in the US (1999), the housing bubble in the US (2005) and Germany (2009) and the property bubble in Spain (2005). The graphs also show that the more significant part of unleveraged bubbles arise from the equity market (approximately 80%), while leveraged bubbles arise more from the housing market (approximately 60%).[19]

5.3 The logit model and estimation results

In this section, we investigate how low real interest rates and its interaction with economic growth can predict bubble events. We define the benchmark dependent variables denoting leveraged and unleveraged bubbles. This is a binary variable that equals one the year before a leveraged bubble starts and zero when an unleveraged bubble starts, $B^L_{i,t} \in [0, 1]$. An estimated coefficient reveals the log-odd of having a leveraged bubble rather than an unleveraged one, for a marginal change of the predictor.

The panel spans the period 1947-2016 for $i = 1, ..., 17$ countries. Therefore, our panel logit models the probability

$$P(B^L_{i,t} = 1 \mid \alpha_i, x_{i,t}) = \frac{exp\{\alpha_i + \beta(x_{i,t})\}}{1 + exp\{\alpha_i + \beta(x_{i,t})\}}, \tag{24}$$

where α_i are country fixed effects and $x_{i,t}$ a vector containing macro predictors. The estimation does not include time dummies that would account for heterogeneity in bubbles probability over time.[20] However, since similar bubbles affected many countries in the sample, we include robust standard errors clustered at the annual level that account for potential correlation in the error terms.

To measure the model's classification ability, we report the area under the receiver operating characteristics curve (AUROC) for each specification. The ROC measures the optimal balance between the true positive and the false positive rates, and therefore the AUROC is the probability that a randomly chosen realization $B^L_{i,t} = 1$ is ranked higher than a randomly chosen $B^L_{i,t} = 0$.[21]

The specifications of the model reported in Table 1 focus on the real interest rate. Model 1 includes country fixed effects and the real interest rate. Model 2 includes fixed effects, the real rate, and its interaction with a dummy accounting for periods in which $r \leq g$, that is the general condition for the leveraged bubble existence arising from our theoretical model (LBC). Model 3 adds the interaction with a dummy accounting for periods in which $r \leq 0$ (LIR). The coefficients in a logit regression are log-odds ratios and negative values mean that the odds ratio is smaller than 1, i.e., a reduction in the probability that the considered event happens. Therefore,

[19] We explore further those different sources of bubbles through robustness checks in Appendix B.1.

[20] Though time dummies would improve the ex-post fit of the model by catching the global time factors that drive the left-hand-side of the logit, they are unknown ex-ante, so they add little help to the out-of-sample forecasting (Schularick and Taylor, 2012).

[21] An AUROC of 0.5 indicates that the ability of the model in classifying realizations is like flipping a coin, whereas a value of 1 indicates a perfect classifier.

the estimated negative log-odds ratios indicate that decreases in the real rate would favor the probability that a coming bubble is leveraged. Instead, the LIR's log-odds ratio is positive as the real rate takes the negative sign and the dummy LIR equals one. Hence, in this case, a positive log-odds ratio can be interpreted in the same way. However, log-odds ratios of those models are not significant.

Table 1: Benchmark logit model for leveraged bubbles.

	(1) r	(2) LBC	(3) LIR	(4) Controls	(5) $(r-g)$	(6) LIR	(7) Controls
Real Rate†	-0.04	-0.03	-0.05	-0.31**	-0.06	-0.03	-0.02
	(0.10)	(0.10)	(0.11)	(0.15)	(0.06)	(0.06)	(0.06)
Real Rate × LBC		-0.13	-0.15	-0.32*			
		(0.12)	(0.13)	(0.19)			
Real Rate† × LIR			0.57	1.90**		-0.31**	-0.38**
			(0.83)	(0.93)		(0.14)	(0.16)
GDP growth				-0.06			
				(0.12)			
Inflation Rate				0.11			0.01
				(0.11)			(0.09)
Total Loans growth				0.12			0.01
				(0.08)			(0.07)
Money (M1) growth				0.15*			0.09
				(0.09)			(0.07)
Stock Price growth				0.01			0.02
				(0.02)			(0.02)
House Price growth				0.08			0.04
				(0.05)			(0.05)
Pseudo R-squared	0.08	0.08	0.09	0.22	0.08	0.10	0.17
AUROC	0.69	0.68	0.68	0.80	0.69	0.72	0.78
Observations	94	94	94	85	94	94	85

Note: Robust standard error are clustered at annual level, country fixed-effects and constant terms are not reported. Apart from the real rate, all the variables are in annual growth rates. (†) Models 5-7 include the real rate minus the GDP growth rate $(r-g)$ in place of the real rate.

Model 4, our benchmark specification, includes macro controls, such as the inflation rate, the growth rates in per-capita terms of real GDP, real loans, real money (M1), real stock and house prices. All the log-odds ratios of interest turn significant at 5% level. A lower real rate implies a significant increase in the odds ratio of having a leveraged bubble rather than an unleveraged, of a 0.73 factor when $r > g$, of a 1.45 when $r \leq g$, and of an 8.1 when $r \leq 0$.[22] Moreover, the inclusion of controls improves the general fit and the model's predictive ability (AUROC passes from 0.68 to 0.8).

Our benchmark model 4 does address a potential omitted-variable bias, i.e., the true effect on the probability of having a leveraged bubble being mitigated by some interaction of the real rate with an omitted macro variable, though none of the latter is significant. This might be due to the relation between r and g highlighted is Section 4.2. Therefore, we explore the condition of

[22]Notice that the odd ratios are obtained by exponentiating the log-odds ratios.

a leveraged bubble, by considering the real rate and the economy's growth rate together. Models 5-7 include $(r-g)$ in place of r, so that we can exclude the interaction with the LBC dummy and control for the combined effect with LIR. The latter's log-odds ratio of the latter is negative and significant in both specifications with and without controls (Models 6-7). Notice that, among the considered observations, if $r \leq 0$ than $(r-g)$ is negative too.[23] This means that further reductions in r would make $(r-g)$ more negative, so that the overall effect of $(r-g)\times$LIR on the probability of having a leveraged bubble is positive.

Excess of saving and a loosening monetary policy drive the economy in an LIR equilibrium. In this equilibrium, agents may rationally buy bubbly assets. However, for the unleveraged bubble to existing, a negative real interest rate is not a sufficient condition. As shown in Section 4.2, only when borrowers can pledge the bubble in the credit market, a negative real rate becomes a sufficient condition. In our empirical strategy, by ex-ante conditioning leveraged bubbles being accompanied by credit-to-GDP overhangs, we mimic a situation in which $\phi = 1$, i.e., bubbles can be fully collateralized, and find evidence that negative real rates predict that leveraged bubbles are more likely than unleveraged ones. However, the real rate does not determine the emergence of the bubble by itself. If we use the same logistic specification of our benchmark model 4 to distinguish normal/no-bubbly periods from bubbly ones, we find the same sign but not significant log-odds ratios.[24]

What it is crucial instead is the risk-shifting mechanism that triggers leveraged bubbles. We have highlighted this feature by assuming that borrowers use the bubble as collateral, and lenders invest in bonds that guarantee a higher return than the bubble and are *risky* because guaranteed by the bubble itself. We test on whether this difference emerges empirically by augmenting our benchmark model by real returns on *risky* and safe assets. We borrow a set of real rates from Jordà et al. (2019): a "long term" real rate is a yield on a 10-years government bond, a "bond" real rate is a total return on a representative basket of long-term government bonds, an "equity" real rate, coming from equity returns (mostly obtained from representative stocks weighted by market capitalization), an "housing" real rate, where returns are obtained from historical house prices and rental indexes, and a "wealth" real rate, in which the nominal rate is a composite rate of safe assets, risky assets, and aggregate wealth, as weighted averages of the individual asset returns.[25]

The results are reported in Table 2. The inclusion of the long-term real rate does change any result of the benchmark model 4 in Table 1. More risky real returns do play an independent role in discerning the nature of the bubble instead. For instance, a marginal increase in the real return on equity does increase the probability that the coming bubble will be leveraged rather than unleveraged of a factor of 1.8. This result is consistent with the theoretical intuition that

[23] Only in two cases g is negative when $r \leq 0$: one is in 1999 in Japan, when, in the middle of the recession following the 1991's bubble crash, the NIKKEI climbed. The second one is in 1974 in Switzerland, during the recession due to the energy crisis.

[24] We find in this case that returns from risky assets do increase the probability of having a bubble. Some results are in Appendix B.1.

[25] See Jordà et al. (2019) for details.

Table 2: Augmented logit model for leveraged bubbles.

	(4) **Benchmark**	(9) **Long rate**	(10) **Bond**	(11) **Equity**	(12) **Housing**	(13) **Wealth**
Real Rate	-0.31**	-0.31	-0.33**	-0.32**	-0.32**	-0.32**
	(0.15)	(0.23)	(0.15)	(0.16)	(0.15)	(0.15)
Real Rate × LBC	-0.32*	-0.32*	-0.42*	-0.43*	-0.42*	-0.42*
	(0.19)	(0.19)	(0.24)	(0.25)	(0.24)	(0.25)
Real Rate × LIR	1.90**	1.90**	2.65**	2.55**	2.64**	2.63**
	(0.93)	(0.94)	(1.29)	(1.28)	(1.30)	(1.30)
Risky return		-0.00	0.60*	0.63*	0.58*	0.62*
		(0.20)	(0.33)	(0.35)	(0.33)	(0.34)
Pseudo R-squared	0.22	0.22	0.26	0.27	0.26	0.27
AUROC	0.80	0.80	0.81	0.82	0.81	0.82
Observations	85	85	77	77	77	77

Note: Robust standard error are clustered at annual level. Macro controls, country fixed-effects and constant terms are not reported.

the access to defaultable debt contract induces risk-loving behavior among agents (Jensen and Meckling, 1979; Stiglitz and Weiss, 1981), which is embedded in our theoretical framework as the essential mechanism distinguishing leveraged from unleveraged bubbles.

Another insight from the exercise is that the log-odd ratios of the interaction of the real rate with LIR (and LBC) increase in terms of magnitude and significance.[26] When risky returns are higher and the risk-free real rate is in the negative territory, leveraged bubbles are way more likely than unleveraged. To summarize, by including returns from risky assets in the benchmark estimation, we restore the link between the assumption and the implication coming from our theoretical framework, and this seems to be supported by the empirical evidence.

6 Conclusions

Notwithstanding our empirical definition of leveraged bubble associates asset price boom-bust cycles with credit overhangs, credit does not play any role in predicting the bubble nature. To discern the bubble's nature, real returns on safe and risky assets, and the former's relationship with the growth rate of the economy seem to play a pivotal role instead. In particular, leveraged bubbles seem to find fertile ground as the real rate approaches to the negative territory.

The theoretical explanation of these empirical findings relies on *risk-shifting*, as emphasized by our OLG model. An economy's growth rate higher than the risk-free real interest rate is a sufficient condition for leveraged bubbles to emerge but not for unleveraged bubbles. Indeed, by financing the bubbly investment via credit, investors/borrowers shift the bubble bursting risk to lenders, and so a relatively low rate of return on bubbly assets makes leveraged bubbles profitable, unlike leveraged ones. Therefore, a negative real interest rate discourages investment

[26] Those results, including the benchmark, are magnified for a shorter sample starting in 1975. This seems to be caused by the exclusion of hyper-inflation periods that drove the real rate in the negative territory.

in risk-free assets and encourages leveraged investment in bubbly assets. In contrast, this is not necessarily the case for an unleveraged bubbly investment, which could be less profitable than risk-free assets, because of the bubble bursting risk, even if the real interest rate is negative.

In an economic scenario in which low risk-free interest rates, and in particular a negative real rate, persist and growth is stagnant, our results are enlightening for monetary authorities in setting their strategies and, in general, should be interpreted as early warning signals for financial stability. Moreover, they point to closer coordination between monetary and macroprudential authorities, which should mitigate the pronounced risk of leveraged bubbly episodes associated with low risk-free interest rates.

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Appendix

A Theoretical Model

A.1 LIR Equilibrium: Steady State

Before listing all the steady state variables, it is worthy to note that equation (14) can be alternatively expressed as

$$Y_{AD} = \left[\frac{(1-\chi)}{1-(1-\chi)(1-\alpha)-\varepsilon\alpha}\right]\left(\frac{1+\beta}{\beta}\right)D\Pi.$$

This alternative expression can be obtained by using equations (1), (2), the definition of profit, the equation

$$\frac{L_t^B}{L_t} = (1-\chi)\frac{\bar{L}^B}{\bar{L}} = \frac{\varepsilon}{(1-\chi)}$$

declared in the main text (footnote 4), and the equation describing the income of borrowers Y^B.

$$p^b = b^B = b^L = 0$$

$$\xi = h = 0$$

$$1+i = 1$$

$$\Pi = \gamma\overline{\Pi}$$

$$1+r = \frac{1}{\gamma\overline{\Pi}}$$

$$Y = Y_{AD} = \left[\frac{(1-\chi)}{1-(1-\chi)(1-\alpha)-\varepsilon\alpha}\left(\frac{1+\beta}{\beta}\right)\gamma\overline{\Pi}D\right]$$

$$L = Y^{\frac{1}{\alpha}} = \left[\frac{(1-\chi)}{(1-\alpha)\chi+\alpha(1-\varepsilon)}\left(\frac{1+\beta}{\beta}\right)\gamma\overline{\Pi}D\right]^{\frac{1}{\alpha}}$$

$$L^B = \frac{\varepsilon}{(1-\chi)}L$$

$$L^L = \frac{1-\varepsilon}{\chi}L$$

$$\frac{W}{P} = \alpha Y$$

$$\frac{Z}{P} = (1-\alpha) Y$$

$$d^B = \gamma \overline{\Pi} D$$

$$d^L = \frac{(1-\chi)}{\chi} \gamma \overline{\Pi} D$$

$$C_y^B = \left[\frac{1}{1-(1-\chi)(1-\alpha)-\varepsilon\alpha}\right]\left(\frac{1+\beta}{\beta}\right)\gamma\overline{\Pi}D - \frac{1}{\beta}\gamma\overline{\Pi}D - T$$

$$C_o^B = T - D$$

$$C_y^L = \frac{1}{\beta}\frac{(1-\chi)}{\chi}\gamma\overline{\Pi}D$$

$$C_o^L = \frac{(1-\chi)}{\chi} D$$

A.2 DNWR à la Eggertsson et al. (2019)

We change slightly the model outlined in Section 3 by assuming the following DNWR

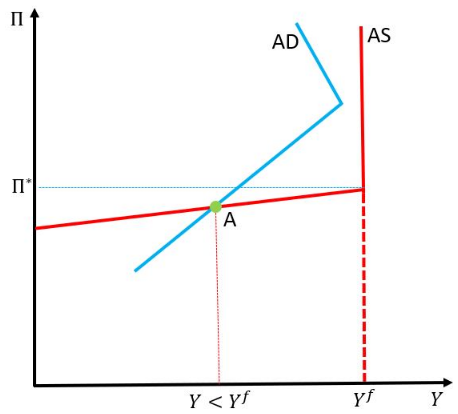
$$W_t = \max\left\{\gamma\Pi^* W_{t-1} + (1-\gamma)\,\alpha P_t \bar{L}^{\alpha-1}, \alpha P_t \bar{L}^{\alpha-1}\right\}, \tag{25}$$

in which the minimum wage is the weighted average of the past nominal wage indexed to the gross inflation target and the flexible wage corresponding to full employment (Eggertsson et al., 2019). The benchmark model is unaffected by this new assumption, apart from the aggregate supply. While the aggregate supply is still expressed by $Y_{AS} = Y^f$ for $\Pi \geq \Pi^*$, it takes the new shape

$$Y_{AS} = \left[\frac{1-\gamma\frac{\Pi^*}{\Pi}}{1-\gamma}\right]^{\frac{\alpha}{1-\alpha}} Y^f \tag{26}$$

for $\Pi < \Pi^*$. If the inflation rate is lower than the target, the nominal wage cannot equate its market clearing level, which falls below the lower bound in (25), and involuntary unemployment arises, leaving output at a level below its potential. The resulting positive relationship between inflation and output is a consequence of the real wage being too high; as inflation rises, the real wage falls, stimulating labor demand and output. The LIR equilibrium, which arises when (15) holds, results from the intersection of the AD curve and the new AS curve, and it is depicted in Figure 3. Although the segment of the AS curve corresponding to binding DNWR is now

Figure 3: LIR Equilibrium with DNWR à la Eggertsson et al. (2019)



upward-sloping and not flat, the LIR equilibrium does not change qualitatively, and it still features

$$1 + r_{nb}^{f} < 1 + r_{nb} < 1$$

$$i = 0$$

$$Y < Y^{f}$$

$$1 < \Pi = \gamma \Pi^{*} < \Pi^{*}.$$

A.3 Unleveraged Bubbly Equilibrium: Borrowers' Bubble Holdings

Given $\phi = 0$, the borrowers' constraints become

$$C_{y}^{B} = Y^{B} + d^{B} - p^{b} b^{B} - T$$

$$C_{o}^{B} = \begin{cases} T - (1+r)\, d^{B} & bubble \quad bursts \\ T + p^{b} b^{B} - (1+r)\, d^{B} & bubble \quad survives \end{cases}$$

$$d^{B}(1+r) = D.$$

The optimality condition of the borrowers' maximization problem are

$$\frac{1}{C_{y}^{B}} p^{b} \geq \beta (1-\rho) \left[\frac{1}{T + p^{b} b^{B} - (1+r)\, d^{B}} \right] p^{b} \tag{27}$$

$$\lambda_{d}^{B} = \frac{1}{C_{y}^{B}(1+r)} - \beta \left[\rho \frac{1}{T - (1+r)\, d^{B}} + (1-\rho) \frac{1}{T + p^{b} b^{B} - (1+r)\, d^{B}} \right] > 0, \tag{28}$$

where λ_d^B is the Lagrange multiplier associated with the credit constraint, and it is strictly positive because borrowers are credit constrained. When borrowers cannot use the bubble as collateral, they buy it if its marginal cost on the left-hand side of (27) is equal to its marginal benefit on the right-hand side, and so equation (27) holds with equality. We assume here borrowers hold the bubble. Equally, lenders hold the bubble, and their optimality conditions are (16) and (17). Combining the optimality conditions of lenders and borrowers yields an upper bound on the optimal demand for bubbles from borrowers:

$$b^B < \frac{T - D}{T(1-\chi)}.$$

Although the borrowers' bubble purchases can be meager for a small value of T, they can be positive, except for $T = D$.

A.4 Leveraged Bubbly Equilibrium: Lenders' Bubble Holdings

We aim to prove, by contradiction, that if borrowers hold the bubble, lenders do not. For $\phi = 1$, the optimality conditions of lenders are still given by (16) and (17), as stated in the main text. For the sake convenience, we rewrite these equations here:

$$\frac{1}{C_y^L}p^b = \beta(1-\rho)\left[\frac{1}{p^b b^L + (1+r)\, d^L}\right]p^b$$

$$\frac{1}{C_y^L} = \beta(1+r)\left[\rho\frac{1}{(1+r)\, d^L} + (1-\rho)\frac{1}{p^b b^L + (1+r)\, d^L}\right].$$

We assume lenders demand a positive quantity of bubbly asset and so the first condition holds with equality. Combining these two equations yields

$$(1+r) = (1-\rho) - \frac{\rho p^b b^L}{d^L} < 1.$$

However, if borrowers hold the bubble too, combining their optimality conditions (19) and (20) yields

$$(1+r) = 1.$$

This leads to a contradiction. Therefore, if borrowers hold the bubble, lenders optimally choose to do not for $\phi = 1$.

B Data and figures

Variable	Obs	Mean	Std.Dev.	Min	Max
GDP (nominal)	1190	4520000	20900000	1.352	1.86E+08
Consumer Price Index	1190	74.469	56.624	2.016	220.082
Total Loans	1186	5030000	28300000	0.184	3.11E+08
Short-term interest rate	1175	5.381	4.073	-2	21.273
Long-term interest rate	1188	6.494	3.61	-0.14	21.503
Bill rate	1100	0.053	0.04	-0.02	0.213
Stock price	1139	450.317	1392.982	0.144	14706.5
House price	1081	95.355	99.773	0.069	569.437
Equity total return	1103	0.13	0.258	-0.884	1.67
Housing total return	1063	0.123	0.102	-0.234	1.363
Wealth total returns	1062	0.116	0.091	-0.147	1.144

Table 3: Summary of the variables. Data can be retrieved from Jordà et al. (2019).

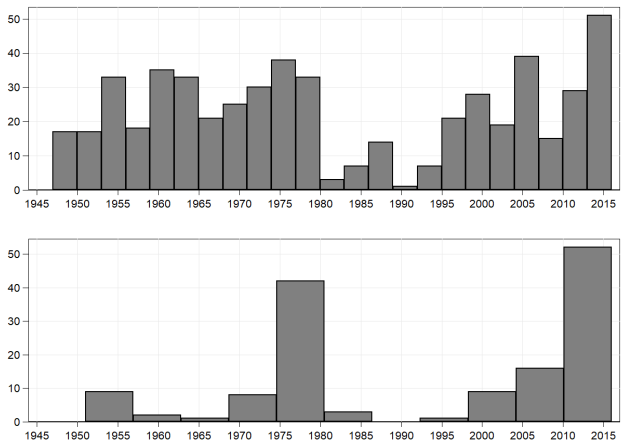


Figure 4: The upper panel shows the frequency of $r \leq g$ periods in the years of the sample and the bottom panel the $r \leq 0$ frequency.

B.1 Robustness checks

Table 4 shows the ability of the real rate on *risky* assets and $(r - g)$ to distinguish leveraged and unleveraged bubbles from normal periods. The overall effect of the two is positive and significant in predicting leveraged bubbles. This result confirms that the latter are associated with higher returns on risky assets. Being the model quite informative in terms of differentiating normal from leveraged bubble periods (AUROC is almost 0.8), this version with risky returns can be a real-time guideline for early warning signals.

Lastly, in Table 5, we propose a different version of models 4-7 by distinguishing leveraged and unleveraged bubbles in bubbles coming from equity and housing markets. Models 18-21

	Leveraged (10)	Unleveraged (11)	Leveraged (12)	Unleveraged (13)	Leveraged (14)	Unleveraged (15)	Leveraged (16)	Unleveraged (17)
	Bill		**Equity**		**Housing**		**Wealth**	
Real Interest Rate	0.28***	0.08	0.27***	0.06	0.30***	0.08	0.29***	0.08
	(0.09)	(0.11)	(0.10)	(0.11)	(0.09)	(0.11)	(0.09)	(0.11)
r-g	-0.20***	-0.08	-0.20***	-0.08	-0.19***	-0.08	-0.19***	-0.08
	(0.07)	(0.08)	(0.07)	(0.08)	(0.07)	(0.07)	(0.07)	(0.07)
Pseudo R-squared	0.13	0.04	0.13	0.04	0.13	0.04	0.13	0.04
AUROC	0.78	0.66	0.78	0.66	0.78	0.68	0.78	0.68
Observations	1012	1012	1012	1012	991	991	991	991

Table 4: Different definitions for the real rate: we take return from bill, equity, housing and wealth. Macro controls and fixed-effects are not reported in the table.

corroborate our previous results with additional information: the log-odd of both r and $(r-g)$ reach the highest magnitude and significance in predicting leveraged housing bubbles (Model 20). In this case, the log-odd of $(r-g)$ is higher than the one of r, meaning that when $r < g$ further reduction of r would increase the probability of having a leveraged housing bubble.

	(18) Lev. equity	(19) Unl. equity	(20) Lev. housing	(21) Unl. housing
Real Interest Rate	0.18	0.15	0.26**	-0.00
	(0.12)	(0.10)	(0.12)	(0.07)
r-g	-0.20***	-0.13	-0.27**	0.00
	(0.06)	(0.09)	(0.12)	(0.06)
Real GDP	-0.01	0.02	0.07**	-0.01
	(0.02)	(0.02)	(0.03)	(0.03)
CPI Inflation	0.03	-0.14**	0.18***	0.13
	(0.08)	(0.07)	(0.06)	(0.08)
Real Money	0.00	-0.01	-0.06***	0.00
	(0.01)	(0.01)	(0.02)	(0.02)
Real Total Loans	0.01	-0.00	0.03***	0.00
	(0.01)	(0.01)	(0.01)	(0.01)
Pseudo R-squared	0.06	0.07	0.20	0.04
AUROC	0.72	0.72	0.86	0.68
Observations	962	1094	1028	660

Table 5: Dependent variables referred to leveraged and unleveraged bubbles are distinguished in equity and housing prices bubbles.

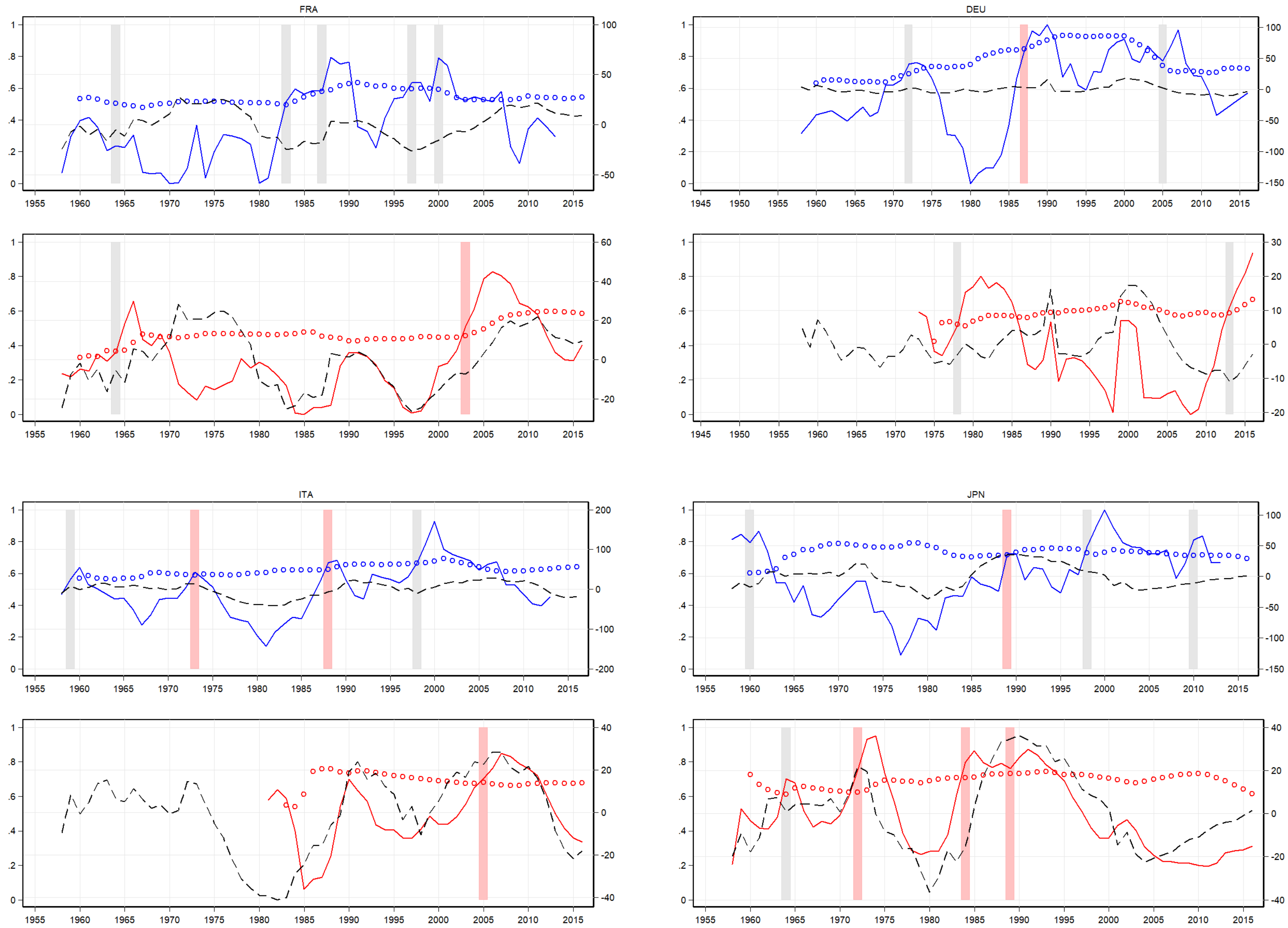


Figure 5: Equity price cycle (blue) and house prices (red), their relative boom-thresholds (dotted) against credit cycle (black dashed). Leveraged bubbles (grey bars) and unleveraged (red bars).

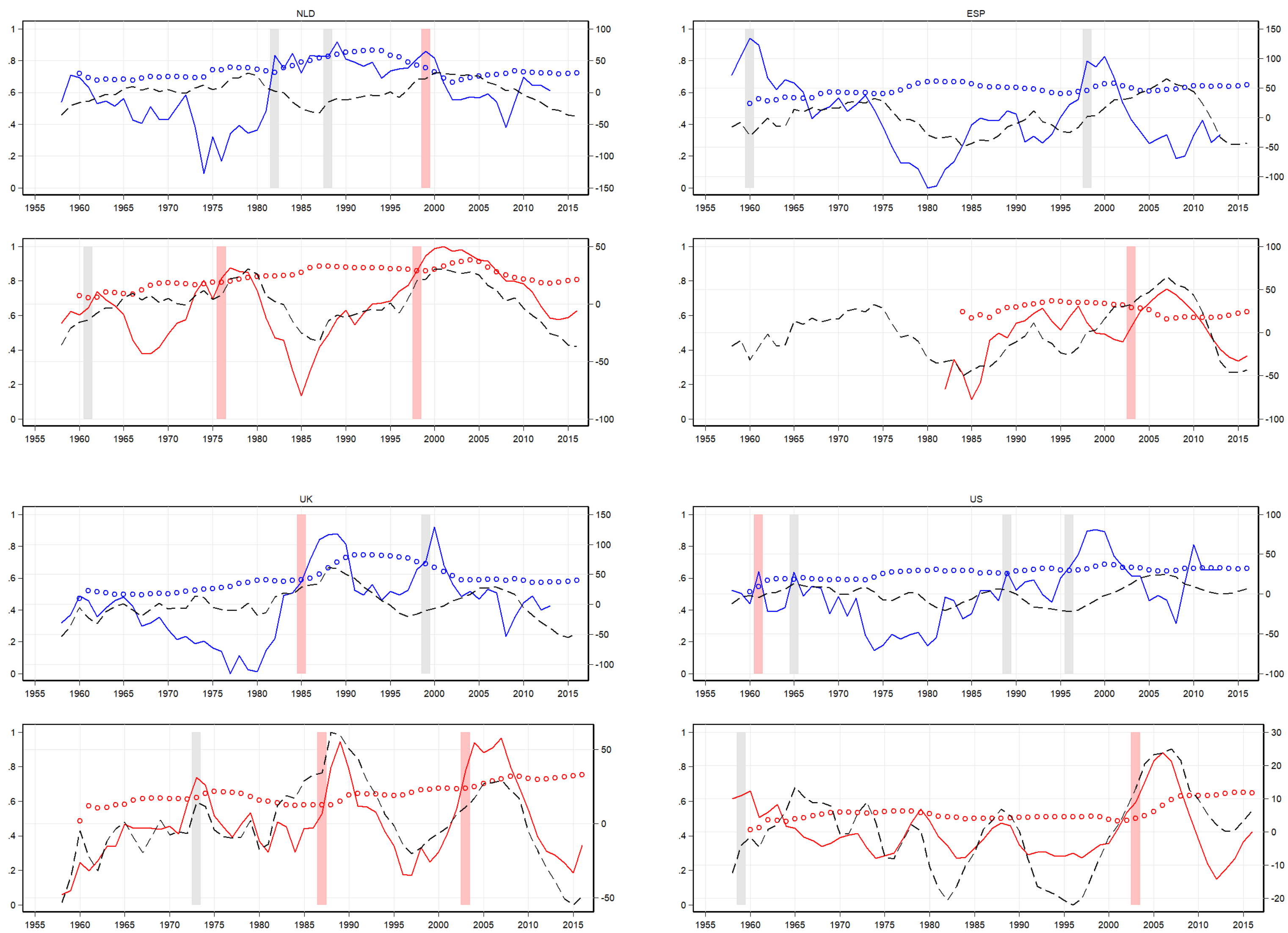


Figure 6: Equity price cycle (blue) and house prices (red), their relative boom-thresholds (dotted) against credit cycle (black dashed). Leveraged bubbles (grey bars) and unleveraged (red bars).