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On the Positive Relation between the Wage Share and Labor Productivity Growth with Endogenous Size and Direction of Technical Change

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Abstract

This paper combines induced innovation and endogenous growth to investigate two issues: the relation between the wage share and labor productivity growth and the potential influence of the saving rate on the steady state wage share. We assume that myopic competitive firms choose the size and direction of technical change to maximize the growth rate of profits. First, we find a condition on the innovation possibility set sufficient to ensure that labor productivity growth is a positive function of the wage share. Second, we show that the steady state wage share depends on the saving rate if, and only if, R&D investment affects the marginal rate of transformation between labor and capital productivity growth. Both results have important policy implications as they clarify under what conditions any factor affecting the wage share or the saving rate will have an impact on labor productivity growth or steady-state income distribution.

Keywords: Induced innovation, R&D, Factors income shares, Growth models

JEL Classification: D24, E25, D33, O30, O41

1 Introduction

Since the early stages of political economy income distribution has been central in the discussion of economic growth and technical change. British Classical economists thought that the accumulation of capital would be financed out of profits, so that they looked at the profit rate as the ultimate regulator of output and capital growth. They also understood the importance of profits in eliciting innovations, as competing capitalists would try to introduce cost-reducing techniques of production in order to earn above average profit rates. In similar fashion, Schumpeter (1911[2008], 1942) argued that technical change is the source of temporary monopolistic profits, and that their existence is essential to provide the necessary incentives for innovation. The Schumpeterian insights have become the foundation of the endogenous growth literature, which developed during the 1990s (see Segerstrom et al., 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992). By introducing the monopolistic competition framework into general equilibrium models, this literature established a positive causal relation between the size of monopolistic profits accruing to innovators and the amount of resources invested to produce technical change (R&D investment).

While all these lines of thinking emphasize the importance of profits in fostering productivity growth, the notion that high real wages or real wage growth may spur labor productivity growth is also well-established both in economic theory and in economic history. The Habakkuk hypothesis (Habakkuk, 1962) maintains that in the nineteenth century the pace of labor-saving technical change was faster in the United States than in Britain because of scarcer and more expensive labor. Allen (2009) singled out the high price of labor relative to energy costs as one of the fundamental forces that triggered the British industrial revolution.

From a theoretical standpoint, this connection is rooted in the incentive to introduce labor-saving innovations for competitive, profit-maximizing, firms that face high labor costs. It has been formally developed and investigated within different analytical frameworks. The theory of induced technical change traces back to Hicks's conjecture that 'a change in the relative prices of the factors of production is itself a spur to invention....directed to economizing the use of a factor which has become relatively expensive' (Hicks 1932, p.124). This result was later proved independently by Kennedy

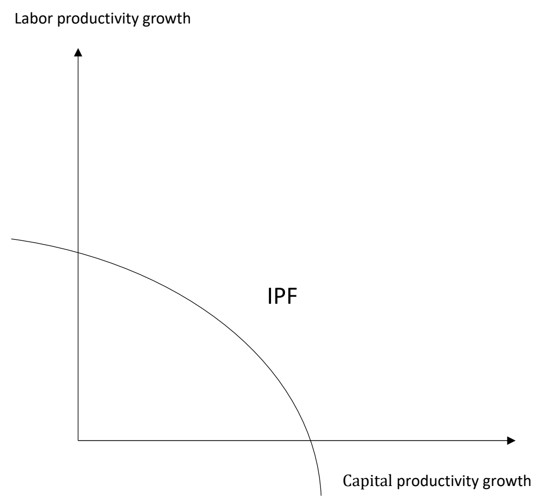

Figure 1: Innovation Possibility Frontier

(1964) and von Weizsacker (1962). They assumed the existence of an innovation possibility frontier (IPF hereafter), which describes the trade-off between freely available capital- and labor-augmenting innovations. As shown in Figure 1, the IPF is decreasing and strictly concave so that substituting capital- to labor- saving technical change becomes progressively harder as capital productivity growth increases.

Competitive firms choose a point on the IPF, that is the *direction* of technical change, in order to maximize the rate of unit cost reduction, or equivalently the growth rate of the profit rate, given levels and prices of labor and capital employed. The firms' optimal choice produces a relation between the direction, or bias, of technical change and functional income distribution: labor- (capital-) productivity growth becomes a positive function of the wage (profit) share. At the macroeconomic level, the mechanism of induced innovation, also known as induced innovation hypothesis (Funk, 2002), has been implemented both in neoclassical (Drandakis and Phelps, 1965; von Weizsacker 1966) and Classical (Shah and Desai, 1981; van der Ploeg, 1987; Foley, 2003; Julius, 2005) growth models with exogenous labor supply. One important implication of these models concerns long-run income distribution. In steady state, the wage share only depends on the shape of the innovation possibility frontier; it is 'exogenous' in the sense that changes in the economy's saving preferences do not affect it. In particular, the curvature of the IPF at the point where capital productivity growth is zero uniquely determines the long-run level of the wage share.

The same positive relation between the wage share and labor productivity growth can be found in a recent literature, which has introduced endogenous, costly, technical change in Classical models of growth. In these contributions (Foley et al. 2019, Ch.9; Tavani and Zamparelli, 2021), competitive firms choose the intensity, or *size*, of technical change rather than its direction. In fact, capital productivity is fixed and firms can only augment the productivity of labor. Specifically, they need to decide how to allocate resources between the alternative uses of physical capital accumulation and labor-saving R&D investment. In this context, a higher wage share makes R&D investment relatively more profitable so that firms divert funds from physical capital to R&D investments thus raising labor productivity growth. Contrary to the induced innovation theory, the saving rate affects long-run income distribution in Classical growth models with endogenous intensity of technical change and exogenous labor supply. In this framework, the wage share is not constrained by the slope of the IPF when capital productivity growth is zero and it will adjust to balance the warranted and the natural growth rate, both of which are affected in different ways by the saving rate. In Tavani and Zamparelli (2021), a higher propensity to save raises capital accumulation (the warranted growth rate) more than labor productivity growth (the natural growth rate): the wage share increases as a result of higher labor demand relative to its fixed supply.

This paper offers a synthesis of induced and endogenous technical change to investigate both the relation between the wage share and labor productivity growth and the long-run determinants of the wage share. We assume that the set of capital- and labor- saving innovations is not exogenously given to firms but depends on the amount of R&D investment they perform. For any given level of R&D investment there will be a distinct IPF, while higher levels of R&D increase the size of the innovation set by pushing the IPF outward. In line with the induced innovation tradition, we assume that firms maximize the instantaneous rate of growth of profits subject to the innovation technology set. In maximizing their objective function they simultaneously choose the allocation of funds between capital accumulation and R&D investment, which determines the size of technical change, and whether to direct technological progress relatively more toward capital- or labor- saving innovations, the direction of technical change. This integration is relevant because the emerging relation between the wage share and labor productivity growth is not necessarily positive, contrary to both the literatures we reviewed, and because it opens up the possibility that the saving rate affect the wage share in steady state within the induced innovation framework.

Specifically, we make the following two contributions. First, we find a technological sufficient condition for a positive relation between the wage share and labor productivity growth. This occurs when the direction of technical change does not affect the marginal productivity of R&D investment and it implies that the wage share affects separately the optimal direction and size of technical change. Secondly, we embed our microeconomic analysis into a Classical growth model with exogenous labor supply. We show that the saving rate affects the long-run distribution of income if, and only if, R&D investments change the marginal rate of transformation between labor and capital productivity growth, which happens when the elasticities of labor and capital productivity growth to R&D investments are different. Since there are no obvious reasons to believe that returns to R&D expenditure are different along the capital or labor productivity dimensions, this result can be read as a generalization of the original induced innovation literature conclusion that the long run labor share is a mere function of the slope of the IPF and thus independent of the saving rate.

Both results have relevant policy implications. On the one hand, any change in labor market institutions which affects the wage share may have an indirect effect on labor productivity growth. On the other hand, a fiscal policy reform that influences an economy's propensity to save could also have long-run distributional consequences.

At this stage, it is useful to anticipate the intuition of our results. With respect to the first one, we should notice that the overall effect of the wage share on labor productivity growth depends on the way it affects both the direction of technical change and the amount of R&D investment a firm decides to perform. A rise in the wage share always makes it more convenient to direct technical change towards labor productivity growth in order to save the more expensive factor of production. The wage share effect on R&D investment, however, is not as straightforward. On the one hand, it is always true that a higher wage share makes increasing labor productivity more convenient than raising the physical capital stock, thus creating an incentive for higher R&D investment. However, when the problem of choosing the size and direction of technical change does not decompose, the benefit of investing in R&D depends on the direction of technical change. This introduces an additional indirect influence of the wage share on the incentive to R&D investment, which may go in the opposite direction and offset the positive direct effect.

Comparing our steady state analysis with the findings of the original induced innovation literature and of the recent Classical growth models with endogenous technical change sheds light on the rationale of our second result. When technical change is exogenous and costless, the steady state wage share is independent of the saving rate as it is determined by the slope of the IPF, or the marginal rate of transformation between labor and capital productivity growth. When technical change is endogenous, both the size and the shape of the innovation set depend of R&D investment. The saving rate influences long-run income distribution if its effect on R&D investments also changes the marginal rate of transformation between labor and capital saving innovations, which still determines the equilibrium wage share. Therefore, the effect of the saving rate purely depends on its potential impact on technology. This mechanism is quite different from the way the saving rate affects income distribution in Classical growth models with endogenous technical change, where changes in the wage share depend on the way shocks to the saving rate impact labor market tightness.

At the onset of the induced innovation literature, a number of contributions have investigated the simultaneous choice of direction and intensity of technical change. Kamien and Schwartz (1969) explored the problem from a microeconomic point of viewof a competitive firm. Nordhaus (1967) solved the infinite horizon problem of a benevolent planner who maximizes the discounted value of consumption per capita. von Weizsacker (1966) analyzed a competitive two-sector economy. Their analysis, however, establishes the standard effect of the wage share on the direction of technical change without exploring its overall effect on labor productivity growth (see Kamien and Schwartz 1969, p. 676, eq. 36).

More recently, the joint determination of intensity and direction of technical change has also been analyzed by Acemoglu (2002, 2007, 2010) within the endogenous growth framework based on monopolistic competition developed in the 1990s. He focuses more on the relation between relative factors scarcity, rather than relative factors share, and factors productivity growth. He shows that the factors elasticity of substitution is crucial in determining the sign of this relation. When the elasticity is lower (higher) than one, a scarcer labor supply will favor labor (capital) augmenting innovations. Our contribution shows that even with zero factors elasticity of substitution the relation between labor productivity growth and relative factors shares can be either positive or negative.

This paper is also related to early 2000s literature, which combined endogenous growth and perfect competition. Similar to Hellwig and Irmen (2001), Bester and Petrakis (2003), and Irmen (2005) firms are willing to pay for innovations in order to earn temporary rents, which are eliminated as soon as the new technology becomes public knowledge. Out of these contributions, only Bester and Petrakis (2003) have analyzed the link income distribution and innovations. They assume that capital productivity is fixed and R&D investment can only improve labor productivity growth. They find the latter is a positive function of the unit labor cost, but only in a partial equilibrium framework.

Finally, Zamparelli (2015) has introduced the endogenous direction and intensity of technical change in a Classical growth model with exogenous labor supply. On the one hand, he does not find an explicit relation between labor productivity growth and the wage share; on the other, even though he finds that the saving rate affects the wage share, he does not discuss the technological assumptions necessary for this result.

The rest of the paper is organized as follows. Section 2 presents the microeconomic problem of the firm and derives the relation between the wage share and labor productivity growth. Section 3 analyzes the macroeconomic long-run equilibrium of the model with a specific focus on the connection between the saving rate and the wage share. Section 4 concludes.

2 The Model

2.1 Households and firms

The economy is populated by a fixed number (normalized to one) of identical households, who are endowed with one unit of homogeneous labor (L) and own a certain share of the capital stock (K). Households supply labor inelastically and, if employed, earn the real wage rate w; they also earn profit income on the capital they own. They save a constant fraction (s) of their total income. Since there are no financial markets, aggregate savings are directly employed to either accumulate capital stock or improve the technology of the representative firm.[1]

2.2 Technology

The final good Y is the numeraire and can be used both for consumption and investment in physical capital or R&D. It is produced by using labor and capital in fixed proportions.[2] There is no depreciation. Letting A and B denote, respectively, labor and capital productivity, the production function is

$$Y = \min\{AL, BK\}. \tag{1}$$

The modeling of technological change includes insights from both the induced innovation literature and the endogenous growth theory. As anticipated in the Introduction, the former represented the evolution of technology through an IPF, which states an inverse relation between the freely available maximum growth rates of labor and

[1]The assumption of a representative firm may appear restrictive, but it is equivalent to assuming a fixed number of firms, each of which has access to the same technology and to the same fraction of aggregate savings.

[2]The adoption of a Leontief production function is particularly innocuous in the context of labor- and capital- productivity growth. The 'impossibility theorem', a generally accepted result in production theory (see Diamond et al., 1978), claims that it is impossible to simultaneously estimate the elasticity of substitution and the bias of technical change. In our setting, factors substitution occurs by directing technical change rather than by moving along a smooth production function.

capital productivity. The frontier is decreasing and strictly concave in order to capture the increasing complexity in the trade-off between labor-augmenting and capital-augmenting innovations. On the other hand, the endogenous growth literature (see for example Aghion, 2010) posited that technical change is a costly activity, which requires the investment of physical or human resources. If we let g_x be the growth rate of variable x, we can define an innovation possibility set as

$$P(g_A, g_B, b) \leq 0, \tag{2}$$

where $b \equiv R/Y$ and R is the amount of final good invested in R&D. P represents an innovation technology that uses one input, b, to produce two outputs, labor and capital productivity growth. Efficiency requires firms to choose points on the set boundary, that is points where $P(g_A, g_B, b) = 0$, otherwise they could increase productivity growth at no cost. For a given level of b, say $\bar{b}$, $P(g_A, g_B, \bar{b}) = 0$ implicitly defines a tranformation curve between g_A and g_B, that is the highest achievable level of labor productivity growth for any level of capital productivity growth. In fact, the P set generates a family of innovation possibility frontiers, each indexed by a different level of R&D investment. As shown in Figure 2, where $b_1 > b_0$, higher investments push the frontier up and to the right.

This representation of technology is flexible enough to encompass both exogenous and endogenous growth. Exogenous growth assumes that technical change is available without costs or investment: $P(g_A, g_B, 0) = 0$, with either g_A, g_B or both strictly positive. When growth is endogenous, innovation is costly so that no investment yields zero productivity growth $P(0, 0, 0) = 0$. Finally, notice that the normalization of R&D investment by total output is imposed in order to rule out explosive growth; this is a standard result in endogenous growth models when R&D inputs consist of an accumulable factor such as physical output, and it is typically justified with the increasing complexity of discovering new ideas.

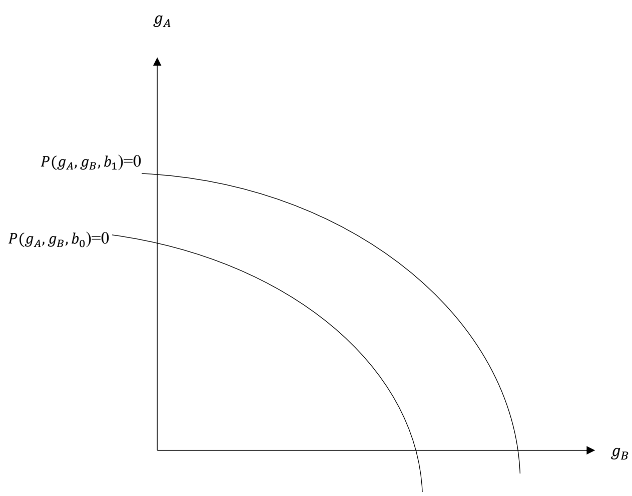

Figure 2: Innovation Possibility Set

In order to make the firm's optimization problem tractable, I generalize the innovation set proposed by Kamien and Schwartz (1969). They assumed that for a given level of R&D spending, the growth rates of productivity growth are related through

$$g_A = f(g_B) = f(\beta), \tag{3}$$

where β is defined by (3) and $\beta, f(\beta) \geq 0$ while $f', f'' < 0$. $f(\beta)$ represents the specific innovation IPF associated to a given level of R&D investment. Different levels of R&D investment push the frontier inward or outward. Contrary to Kamien and Schwartz (1969) we do not restrict these shifts to occur as radial homothetic contractions and expansions. Accordingly we posit

$$g_A = H(f(\beta), b) \tag{4}$$

$$g_B = F(\beta, b), \tag{5}$$

where H and F are twice differentiable and, on the one hand, $H'_b, F'_b > 0$ and $H''_{b,b}, F''_{b,b} < 0$ convey the idea that productivity growth is an increasing and concave function of R&D investment; while, on the other hand, $F'_\beta, H'_f > 0$ imply that factors productivity growth increases when the direction of technical change is biased in their respective direction. We also add $F''_{\beta,\beta} = H''_{f,f} = 0$ so that the direction of technical change raises linearly each productivity growth rate.

Improvements in technology allow innovators to earn instantaneous rents. The new knowledge becomes freely available to all producers immediately after rents are obtained. This assumption represents the classical notion of competition where firms introduce innovations to earn a temporary advantage over their competitors, which disappears as soon as rival firms are able to imitate the new technology. A similar framework is present in Hellwig and Irmen (2001), Bester and Petrakis (2003), and Irmen (2005), who have introduced endogenous technical change into neoclassical perfectly competitive growth models.

2.3 Income distribution, saving allocation and optimal productivity growth

The owners of the representative firm have no incentive to keep the firm operating with spare capacity or to hire unproductive labor, therefore $AL = BK$, so that the number of employed workers in the economy is $L = BK/A$. We denote the wage share as $\omega \equiv wL/Y = w/A$, equal to the unit labor cost. Accordingly, total profits are $\Pi = Y - wL = Y(1-\omega) = BK(1-\omega)$. Savings are spent to accumulate physical capital or as R&D investment. From the standpoint of profit-maximization, the two types of investment pose a trade-off. They both increase total profits. While capital accumulation increases the size of a firm, innovations raise its profits per unit of capital by reducing unit costs. Letting μ be the share of savings invested in R&D, the R&D investment share of output is:

$$b = R/Y = \mu s Y/Y = \mu s. \tag{6}$$

Physical capital accumulation, on the other end, obeys:

$$g_K = (1-\mu)sY/K = (1-\mu)sB. \tag{7}$$

In order to define the objective function of the representative firm, we extend the original proposal by Kennedy (1964). He assumed that firms take input levels and prices as given and choose the direction of technical change to maximize the instantaneous rate of growth of the profit rate. The myopic behavior is justified because the temporary rents due to innovating dissolve instantaneously as the new technology becomes public knowledge. In our setting, the representative firm still acts myopically and takes inputs prices as given, but besides the direction of technical change it also chooses the allocation of savings between R&D investment and capital accumulation in order to maximize the rate of growth of profits.[3] If the firm take the real wage as given, differentiation of total profits with respect to time yields $\dot{\Pi} = \dot{B}K(1-\omega) + \dot{K}B(1-\omega) + g_A \omega BK$, where the time derivative of variable x is denoted by $\dot{x}$. The corresponding rate of growth of profits is

$$g_\Pi = g_B + g_K + g_A \omega/(1-\omega). \tag{8}$$

Substituting from equations (4), (5), (6) and (7), the firms' problem is to choose β and μ so as to maximize $g_\Pi = F(\beta, s\mu) + s(1-\mu)B + H(f(\beta), s\mu)\omega/(1-\omega)$. We study this problem by first assuming two specific functional forms for H and F and later discussing the general case.

2.3.1 Two special cases

Let us start by positing $g_A = f(\beta) + (s\mu)^\alpha$, and $g_B = \beta + (s\mu)^\alpha$, with $\alpha \in (0,1)$. Notice that for any point where $f(\beta) \neq \beta$ the amount of R&D expenditure affects the ratio g_A/g_B. If we denote the optimal level of a choice variable by $*$, the first order conditions with respect to β and μ are

$$-f'(\beta^*) = \frac{1-\omega}{\omega}, \tag{9}$$

and (after some manipulations)

$$\mu^* = \frac{1}{s}\left(\frac{\alpha}{B}\frac{1}{1-\omega}\right)^{\frac{1}{1-\alpha}}. \tag{10}$$

Equations (9) and (10) show that the choice of direction and intensity of technical change decomposes into two parts. Equation (9) demands the equality between the

[3]Notice that the rate of growth of the profit *rate* and the rate of growth of profits coincide when the level of capital stock is given, as originally assumed by Kennedy (1964).

slope of the IPF and of the relative unit factors cost; this is the same exact condition, which produced the positive relation between the wage share and labor productivity growth under the original induced innovation hypothesis. In fact, total differentiation of (9) yields $d\beta^*/d\omega = 1/\left(f''(\beta^*)\,(\omega)^2\right) < 0$: for a given amount of R&D investments (the position of the IPF), a rise in the wage share biases the direction of technical change away from capital productivity growth and in favor of labor productivity growth. Equation (10), on the other hand, shows that R&D investments are a positive function of the wage share because raising productivity growth becomes relatively more profitable than capital accumulation when unit labor costs increase. We can use the optimal values for β and μ to solve for the equilibrium labor productivity growth as

$$g_A^* = f(\beta^*) + \left(\frac{\alpha}{B}\frac{1}{1-\omega}\right)^{\frac{\alpha}{1-\alpha}},$$

which shows that an increase in the wage share unequivocally raises labor productivity growth given $f'(\beta^*)d\beta^*/d\omega > 0$. In other words, both effects of the labor share on the direction and the size of technical change move in the same direction to contribute to labor-saving technical change.

We can now consider the alternative specification $g_A = f(\beta)\,(s\mu)^\alpha$, and $g_B = \beta\,(s\mu)^\alpha$. After some minor manipulations, the first order conditions with respect to β and μ are

$$-f'(\beta^*) = \frac{1-\omega}{\omega}, \tag{11}$$

and

$$\mu^* = \frac{1}{s}\left(\frac{\alpha}{B}\left(\beta^* + \frac{\omega}{1-\omega}f(\beta^*)\right)\right)^{\frac{1}{1-\alpha}}. \tag{12}$$

The system made up of equations (11) and (12) shows that in this case the choice of direction of technical change and size of R&D investment does not fully decompose. Equation (11) is identical to equation (9) and finds the optimal direction of technical change β^* as a negative function of the wage share. On the other hand, equation (12) shows that the optimal size of R&D investment μ^* depends on the wage share both directly and indirectly through its effect on β^*. This has important consequences on the relation between labor productivity growth and the wage share. If we substitute from the two first order conditions into $g_A = f(\beta)\,(s\mu)^\alpha$ we find:

$$g_A^* = f(\beta^*(\omega))\left(\frac{\alpha}{B}\left(\beta^*(\omega) + \frac{\omega}{1-\omega} f(\beta^*(\omega))\right)\right)^{\frac{\alpha}{1-\alpha}}, \qquad (13)$$

where we emphasized the dependence of β^* on the wage share. On the one hand, a rise in the wage share produces a bias in technical change that unequivocally raise labor productivity growth: $f'(\beta^*)d\beta^*/d\omega > 0$. The effect on the size of R&D investment μ^*, on the other hand, is ambiguous. It depends on whether a higher wage share makes investment in productivity growth more profitable than the alternative investment in physical capital. While the higher wage share necessarily increases labor productivity growth and its weight in the objective function ($\omega/(1-\omega)$), it lowers capital productivity growth thus making the overall effect on aggregate productivity uncertain. If the fall in capital productivity growth is strong enough we may see a reduction in the share of R&D investment and a rise in physical capital investment. Notice that this effect is absent in our previous example where $g_A = f(\beta) + (s\mu)^\alpha$ and $g_B = \beta + (s\mu)^\alpha$. In that case, the marginal benefit of investing in productivity does not depend on β^*, which affects the level of capital and labor productivity. This example sheds light on the intuition behind the possibility that a negative relation between the wage share and labor productivity growth may emerge. It requires that the optimal choice of the size of technical change depends on the optimal direction of technical change. In this case, the intensity of technical change depends on the wage share both directly and indirectly through its effect on the direction of technical change. While the direct effect is always positive as it makes investing in productivity growth more rewarding the investing in capital accumulation, the indirect effect may have an opposite sign and even cause a decline in the size of technical change.

2.3.2 The general case

We now generalize our analysis by removing any specific functional form on the two innovation functions, so that $g_A = H(f(\beta), s\mu)$ and $g_B = F(\beta, s\mu)$. In this case, choosing β and μ to maximize $g_\Pi = F(\beta, s\mu) + s(1-\mu)B + H(f(\beta), s\mu)\omega/(1-\omega)$ yields the following system of first order conditions

$$-f'(\beta^*)\frac{H'_f(f(\beta^*), s\mu^*)}{F'_\beta(\beta^*, s\mu^*)} = \frac{1-\omega}{\omega} \tag{14}$$

$$F'_\mu(\beta^*, s\mu^*) + \frac{\omega}{1-\omega}H'_\mu(f(\beta^*), s\mu^*) = B. \tag{15}$$

We are interested in understanding under what conditions this system necessarily produces a positive effect of the wage share on labor productivity growth. Notice that

$$\frac{dg_A^*}{d\omega} = H'_f(f(\beta^*), s\mu^*)f'(\beta^*)\frac{d\beta^*}{d\omega} + sH'_\mu(f(\beta^*), s\mu^*)\frac{d\mu^*}{d\omega}.$$

Since $f' < 0$ and $H'_f, H'_\mu > 0$, a sufficient condition for $\frac{dg_A^*}{d\omega} > 0$ is that $\frac{d\beta^*}{d\omega} < 0$ and $\frac{d\mu^*}{d\omega} > 0$. This condition requires some technological restrictions summarized in the following

Proposition 1. *If* $F''_{\mu,\beta}(\beta^*, s\mu^*) = F''_{\beta,\mu}(\beta^*, s\mu^*)$=$H''_{\mu,\beta}(f(\beta^*), s\mu^*) = H''_{\beta,\mu}(f(\beta^*), s\mu^*) = 0$ *then* $\frac{dg_A^*}{d\omega} > 0$.

Proof. See the Appendix. □

Let us focus on the technological assumption made in Proposition 1. The second-order mixed partial derivatives can be expressed as $\frac{d}{d\beta}\left(\frac{dg_i^*}{d\mu}\right)$ and $\frac{d}{d\mu}\left(\frac{dg_i^*}{d\beta}\right)$, where $i = A, B$. We know from Young's theorem that $\frac{d}{d\beta}\left(\frac{dg_i^*}{d\mu}\right) = \frac{d}{d\mu}\left(\frac{dg_i^*}{d\beta}\right)$. Imposing

$\frac{d}{d\beta}\left(\frac{dg_i^*}{d\mu}\right) = \frac{d}{d\mu}\left(\frac{dg_i^*}{d\beta}\right) = 0$ means that, on the one hand, a change in the direction of technical change does not affect the marginal productivity of R&D investment and, on the other hand, that a shock to the intensity of technical change has no influence on the effect of the direction of technical change on productivity growth. Eliminating these cross-effects between the two choice variables simplifies the analysis greatly. Each of the two first order conditions (14) and (15) individually determines the effect of a shock to the wage share on, respectively the direction and the size of technical change, which ensures $\frac{d\beta^*}{d\omega} < 0$ and $\frac{d\mu^*}{d\omega} > 0$.

This general case can be best illustrated by going back to our two specific examples. Our first case $g_A = f(\beta)+(s\mu)^\alpha$ and $g_B = \beta+(s\mu)^\alpha$ satisfies $F''_{\mu,\beta} = F''_{\beta,\mu} = H''_{\mu,\beta} = H''_{\beta,\mu} = 0$ and it confirms $\frac{dg_A^*}{d\omega} > 0$ always. When, on the other hand, $F''_{\mu,\beta} = F''_{\beta,\mu} \neq 0$ and $H''_{\mu,\beta} = H''_{\beta,\mu} \neq 0$ then $\frac{d\beta^*}{d\omega} > 0$ and $\frac{d\mu^*}{d\omega} < 0$ cannot be simultaneously excluded so that $\frac{dg_A^*}{d\omega} < 0$ is in principle a possibility. In our second example, where $g_A = f(\beta)\,(s\mu)^\alpha$ and $g_B = \beta\,(s\mu)^\alpha$, we have $H''_{\mu,\beta} = H''_{\beta,\mu} = f'(\beta)\alpha s^\alpha \mu^{\alpha-1} < 0$ and $F''_{\mu,\beta} = F''_{\beta,\mu} = \alpha s^\alpha \mu^{\alpha-1} > 0$, which confirms that $\frac{dg_A^*}{d\omega} < 0$ is possible.

3 Income distribution implications

As anticipated in the Introduction, the induced innovation hypothesis has been embedded both in neoclassical and Classical growth models with exogenous labor supply. An important result common to both frameworks is that long-run income distribution depends solely on technology, and specifically on the curvature of the IPF; this implies that the saving rate does not affect the steady state wage share. In this section, we show how the generalization of innovation technology to simultaneously encompass the choice of direction and size of technical change affects the role played by the saving rate in the steady state equilibrium. We illustrate this result by implementing the induced innovation hypothesis into a Classical growth model. Shah and Desai

(1981) offer an example of classical growth where firms choose the direction of technical change, but where innovations can be implemented with no cost. They do so by introducing the IPF into the classical Goodwin's (1967) growth cycle model. The aggregate economy is described by three differential equations, and the output-capital ratio, the labor share and the employment rate are the three state variables (see also Foley, 2003; Julius, 2005). Since according to the original IPF firms do not perform R&D investment, labor productivity growth only depends on capital productivity growth, say $g_A = j(g_B)$, while all savings are invested in physical capital accumulation so that $g_K = sB$. Notice also that when exogenous labor supply is normalized to one the employment rate coincides with total employment L. In our notation, the dynamical system is:

$$-j'(g_B^*) = \frac{1-\omega}{\omega}$$

$$g_L = g_B^* + sB - j(g_B^*)$$

$$g_\omega = g_w - j(g_B^*) = m(L) - j(g_B^*),$$

where $g_w = m(L)$ is a real wage Phillips curve describing the positive effect of labor market tightness on real wage growth. Steady states require that capital productivity growth be turned off, so that $g_B^* = 0$ determines the long run wage share. If we denote steady state values by ss we can find ω_{ss} as solution to $-j'(0) = \frac{1-\omega_{ss}}{\omega_{ss}}$, that is $\omega_{ss} = 1/(1-j'(0))$. The steady state wage share is determined by the slope of the IPF where capital productivity growth is zero, irrespective of the saving rate.

Le us now explore how the dynamical system changes when innovations are costly and require investment, that is when we adopt the innovation technology $g_A = H(f(\beta), s\mu)$ and $g_B = F(\beta, \mu)$. In particular, in order to obtain analytical conclusions, let us slightly modify our second example and assume $g_A = f(\beta)\,(s\mu)^\gamma$ and $g_B = \beta\,(s\mu)^\alpha$, with $\gamma \in (0,1)$. Notice that when $\gamma = \alpha$ we are back to our second example. If, on the contrary, $\gamma \neq \alpha$, R&D investments affect labor and capital productivity growth asymmetrically and the expansion of the innovation possibility frontiers is non-homothetic. In this case, choosing β and μ to maximize $g_\Pi = \beta\,(s\mu)^\alpha + s(1-\mu)B + f(\beta)\,(s\mu)^\gamma\,\omega/(1-\omega)$ yields the following first order conditions

$$-f'(\beta^*)\,(s\mu^*)^{\gamma-\alpha} = \frac{1-\omega}{\omega} \tag{16}$$

$$\beta^*\alpha\,(s\mu^*)^{\alpha-1} + f(\beta^*)\gamma\,(s\mu^*)^{\gamma-1}\,\omega/(1-\omega) = B. \tag{17}$$

As a first point, notice that the left hand side of (16) is the marginal rate of transformation between labor and capital productivity growth. We can calculate it by plugging $\beta = g_B/\,(s\mu)^\alpha$ into $g_A = f(\beta)\,(s\mu)^\gamma$ and finding $-\frac{dg_A}{dg_B} = -f'(\beta)(s\mu)^{\gamma-\alpha}$. Firms choose the optimal direction of technical change by equalizing the slope of the IPF to the relative unit factors cost. The difference with the original induced innovation theory is that the slope of the IPF depends in principle on the size of R&D investment. Next, focus on the system of equations (16) and (17). It implicitly finds β^*, μ^* as functions of the saving rate and the two state variables wage share and output-capital ratio, say $\beta^* = \beta(s,\omega,B)$ and $\mu^* = \mu(s,\omega,B)$. We can use them to define a differential equation for the output-capital ratio as $g_B = \beta(s,\omega,B)\,(s\mu(s,\omega,B))^\alpha$. The rest of the dynamical system is

$$g_L = \beta^*\,(s\mu^*)^\alpha + s(1-\mu^*)B - f(\beta^*)\,(s\mu^*)^\gamma$$

$$g_\omega = m(L) - f(\beta^*)\,(s\mu^*)^\gamma\,.$$

We know that in the steady state capital productivity growth is turned off, that is

$\beta^* = 0$. First notice that if $\gamma = \alpha$, equation (16) finds the steady state wage share independently of the saving rate as $\omega_{ss} = 1/(1 - f'(0))$. This is the original result of the induced innovation hypothesis, where long-run income distribution depends only on the slope of the IPF when capital productivity growth is zero. However, this is not the case when $\gamma \neq \alpha$. Equation (16) yields $\mu_{ss} = (-f'(0))^{\frac{1}{\alpha-\gamma}} \left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1}{\alpha-\gamma}} /s$, while from equation (17) we find $\mu_{ss} = \left(\frac{f(0)\gamma}{B_{ss}} \frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1}{1-\gamma}} /s$. We can use both equations jointly to obtain an isocline in the (ω_{ss}, B_{ss}) space: $B_{ss} = \gamma \frac{f(0)}{-f'(0)^{\frac{1-\gamma}{\alpha-\gamma}}} \left(\frac{1-\omega_{ss}}{\omega_{ss}}\right)^{\frac{1-\alpha}{\alpha-\gamma}}$. If we turn to the law of motion of the employment rate and we set the steady state condition $g_L = 0$ while using $\mu_{ss} = \left(\frac{f(0)\gamma}{B_{ss}} \frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1}{1-\gamma}} /s$, we get an additional isocline in the (ω_{ss}, B_{ss}) plane: $B_{ss} = f(0)\gamma^{\gamma} \left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\gamma} \left(\gamma\frac{\omega_{ss}}{1-\omega_{ss}} + 1\right)^{1-\gamma} /s^{1-\gamma}$, as shown in the Appendix.

The two isoclines jointly determine the long-run values of the wage share and the capital-output ratio. Since the saving rate enters the second isocline both through capital accumulation and through the size of R&D investment, it also affects the steady state wage share. In the Appendix we show that the two isoclines can be used to find ω_{ss} as solution to

$$s = (-f'(0))^{\frac{1}{\alpha-\gamma}} \left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1+\gamma-\alpha}{\alpha-\gamma}} \left(\frac{\omega_{ss}}{1-\omega_{ss}} + 1/\gamma\right).$$

We can now state

Proposition 2. *A rise in the saving rate has a positive, null or negative effect on the steady state wage share depending on whether* $\alpha \gtreqless \gamma$.

Proof. See the Appendix. $\square$

We can interpret this result by looking at equation (16) evaluated at the steady state:

$$-f'(0)\left(s\mu_{ss}\right)^{\gamma-\alpha} = \frac{1-\omega_{ss}}{\omega_{ss}}.$$

It states that the steady state wage share is determined by the marginal rate of transformation between labor and capital productivity growth when capital productivity growth is zero. We have already seen how this implies that the steady state wage share is independent of the saving rate when $\alpha = \gamma$. This occurs because under this condition R&D investments improve labor and capital productivity growth at the same rate, which ensures the homothetic expansion and contraction of the IPF family. Along any ray coming out of the origin, the marginal rate of transformation is constant. It is independent of the amount of R&D investment performed and, in turn, of the saving rate. In particular, this is also true when $\beta = 0$, which is the steady state condition.

On the contrary, the marginal rate of transformation depends on the amount of R&D investment and the saving rate when $\alpha \neq \gamma$. If $\gamma > \alpha$, the marginal rate of transformation is an increasing function of expenditure in R&D. Higher R&D investments make the slope of the IPF steeper and this will require a lower equilibrium wage share. Symmetrically, if $\gamma < \alpha$, the slope of the IPF becomes flatter with more expenditure in R&D, which makes the equilibrium wage share rise in response.

The possibility that the saving rate affect long-run income distribution thus depends on its potential influence on the shape of innovation possibility set. As such, this mechanism is quite different from the way the saving rate affects the wage share in Classical growth models with endogenous technical change, where distributional changes are the results of higher capital accumulation and labor demand relative to the exogenous labor supply. On the contrary, our result appears more in line with the conclusions of the induced innovation literature where the equilibrium wage share is a mere function of the curvature of the IPF. In fact, even when the size of technical change is endogenous, the steady state wage share still only depends on the marginal rate of transformation between labor and capital productivity growth. The saving rate becomes relevant only if R&D investments have different returns on labor and capital productivity growth, that is when $\alpha \neq \gamma$. Assessing this possibility may be an empirical issue and future research on the topic may help verifying it. In the meantime, the more plausible and intuitive assumption is that R&D efforts are equally productive along both the labor and capital direction. From this standpoint, our result appears more like a generalization than a confutation of the original induced innovation hypothesis.

4 Conclusions

Most advanced economies have recently experienced a slowdown in productivity growth (Dieppe, 2021). The notion that declining, or low, real wages may be contributing to this trend is becoming increasingly popular in the public debate: 'Faced with reduced labour costs, employers have lesser incentives to substitute capital for labour, especially in labour intensive sectors, which hinders diffusion of artificial intelligence and other technologies.' (ILO, 2018). More in general, several commentators have suggested that rising income inequality is likely an important factor in explaining the present sluggish level of economy activity known as 'secular stagnation'. This relation may operate both through demand side factors, such as a higher average propensity to save (see for example Summers, 2014; Storm, 2017; and Kiefer et al., 2020), and by means of supply side elements, like the limited incentives to innovate due to low labor costs (Petach and Tavani, 2020).

The simultaneous rise in income inequality and productivity slowdown is also at the center of our paper. We have reviewed different strands of economic literature that, by focusing either on the direction or on the size on innovation, have provided strong microfoundations for a positive relation between the wage share and labor productivity growth. We have found technological restrictions that ensure this relation holds even when firms simultaneously choose both the direction and the size of innovation: if the direction of technical change does not affect the marginal productivity of R&D investment, a rise in the wage share necessarily increases labor productivity growth.

This condition implies that the wage share affects separately the optimal direction and size of technical change. Furthermore, we have shown that the saving rate may have an effects on the steady state wage share, but only if R&D investments change the marginal rate of transformation between labor and capital productivity growth. Since this requires the unintuitive condition that R&D returns be different along the labor and capital dimension, our result appears in line with the original induced innovation literature conclusion that the long-run labor share is a mere function of the innovation technology.

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6 Appendix

6.1 Proof of Proposition 1

If we totally differentiate the system of first order conditions (14) and (15) with respect to β^*, μ^* and ω, after rearranging and dropping the arguments of the function for parsimony, we find

$$\left(f''\frac{H'_f}{F'_\beta}+\frac{f'}{(F'_\beta)^2}\left(H''_{f,f}f'F'_\beta-F''_{\beta,\beta}H'_f\right)\right)\frac{d\beta^*}{d\omega}+s\frac{f'}{(F'_\beta)^2}\left(H''_{f,\mu}F'_\beta-F''_{\beta,\mu}H'_f\right)\frac{d\mu^*}{d\omega}=\frac{1}{\omega^2}$$

$$\left(F''_{\mu,\beta}+\frac{\omega}{1-\omega}H''_{\mu,\beta}\right)\frac{d\beta^*}{d\omega}+s\left(F''_{\mu,\mu}+\frac{\omega}{1-\omega}H''_{\mu,\mu},s\mu^*)\right)\frac{d\mu^*}{d\omega}=-\frac{H'_\mu}{(1-\omega)^2}.$$

Let us focus on the role played by the second-order mixed partial derivatives. From Young's theorem we know $F''_{\mu,\beta}=F''_{\beta,\mu}$ and $H''_{\mu,\beta}=H''_{\beta,\mu}$ and from the chain rule $H''_{\beta,\mu}=H''_{f,\mu}f'$. When all the mixed partial derivatives are null, the system simplifies to

$$\frac{d\beta^*}{d\omega} = \frac{1}{\omega^2 f'' \frac{H_f'}{F_\beta'}} < 0$$

$$\frac{d\mu^*}{d\omega} = -\frac{H_\mu'}{s(1-\omega)^2} / \left(F_{\mu,\mu}'' + \frac{\omega}{1-\omega} H_{\mu,\mu}'' \right) > 0.$$

This shows that the two conditions sufficient for the positive effect of the wage share on labor productivity growth are satisfied, and $\frac{dg_A^*}{d\omega} > 0$ follows necessarily.

6.2 Steady state solution and proof of Proposition 2

Let us start with $g_L = 0$ and $\beta_{ss} = 0$. We have $s(1-\mu_{ss})B_{ss} = f(0)\left(s\mu_{ss}\right)^{\gamma}$. Plugging $\mu_{ss} = \left(\frac{f(0)\gamma}{B_{ss}} \frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1}{1-\gamma}} / s$ into the previous equation and rearranging yields $sB_{ss} = B_{ss}\left(\frac{f(0)\gamma}{B_{ss}} \frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1}{1-\gamma}} + f(0)\left(\frac{f(0)\gamma}{B_{ss}} \frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{\gamma}{1-\gamma}}$,which we can solve for B_{ss} to find $B_{ss} = f(0)\gamma^{\gamma}\left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\gamma}\left(\gamma\frac{\omega_{ss}}{1-\omega_{ss}} + 1\right)^{1-\gamma} / s^{1-\gamma}$.

Next, use the two isoclines in the (B_{ss}, ω_{ss}) space to find: $\gamma \frac{f(0)}{-f'(0)^{\frac{1-\gamma}{\alpha-\gamma}}} \left(\frac{1-\omega_{ss}}{\omega_{ss}}\right)^{\frac{1-\alpha}{\alpha-\gamma}} = f(0)\gamma^{\gamma}\left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\gamma}\left(\gamma\frac{\omega_{ss}}{1-\omega_{ss}} + 1\right)^{1-\gamma} / s^{1-\gamma}$. Simplifying, elevating to the power of $1/(1-\gamma)$ and rearranging yields:

$s = \left(-f'(0)\right)^{\frac{1}{\alpha-\gamma}}\left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1+\gamma-\alpha}{\alpha-\gamma}}\left(\frac{\omega_{ss}}{1-\omega_{ss}} + 1/\gamma\right)$.

We can totally differentiate the previous equation w.r.t. ω_{ss} and s to find: $ds = \left(-f'(0)\right)^{\frac{1}{\alpha-\gamma}}\left\{\frac{1+\gamma-\alpha}{\alpha-\gamma}\left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1+\gamma-\alpha}{\alpha-\gamma}-1}\frac{1}{(1-\omega_{ss})^2}\left(\frac{\omega_{ss}}{1-\omega_{ss}} + 1/\gamma\right) + \left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1+\gamma-\alpha}{\alpha-\gamma}}\frac{1}{(1-\omega_{ss})^2}\right\}d\omega_{ss}$. Hence $ds = \left(-f'(0)\right)^{\frac{1}{\alpha-\gamma}}\left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1+\gamma-\alpha}{\alpha-\gamma}-1}\frac{1}{(1-\omega_{ss})^2}\left\{\frac{1+\gamma-\alpha}{\alpha-\gamma}\left(\frac{\omega_{ss}}{1-\omega_{ss}} + 1/\gamma\right) + \left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)\right\}d\omega_{ss}$, $ds = \left(-f'(0)\right)^{\frac{1}{\alpha-\gamma}}\left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1+\gamma-\alpha}{\alpha-\gamma}-1}\frac{1}{(1-\omega_{ss})^2}\left\{\frac{1}{\alpha-\gamma}\frac{\omega_{ss}}{1-\omega_{ss}} + \frac{1+\gamma-\alpha}{\alpha-\gamma}\frac{1}{\gamma}\right\}d\omega_{ss}$, and finally $ds = \left(-f'(0)\right)^{\frac{1}{\alpha-\gamma}}\left(\frac{\omega_{ss}}{1-\omega_{ss}}\right)^{\frac{1+\gamma-\alpha}{\alpha-\gamma}-1}\frac{1}{(1-\omega_{ss})^2}\frac{1}{\alpha-\gamma}\left\{\frac{\omega_{ss}}{1-\omega_{ss}} + \frac{1+\gamma-\alpha}{\gamma}\right\}d\omega_{ss}$. Since all factors multiplying $d\omega_{ss}$ are positive save for $(\alpha-\gamma)$, we can conclude that $sign\frac{d\omega_{ss}}{ds} = sign\left(\alpha-\gamma\right)$.